Pure spinor fields for supergravity

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Plan

From on-shell superspace to pure spinors — review

- Pure spinors
- Cohomology
- Relation to on-shell superfields
- Non-minimal variables and integration
- Batalin–Vilkovisky actions

$D = 11$ supergravity

- 3-form and vielbein cohomologies
- Linearised action
- Interaction

Gauge fixing

Conclusions

JHEP 01 (2010) 117 [arXiv:0912.1814], 1001.0112 (and earlier)

From on-shell superspace to pure spinors

Maximally supersymmetric models (16 supercharges for gauge theories, 32 with gravity) have on-shell supermultiplets. There is no finite set of auxiliary fields.

Examples:

- $D = 10$ super-Yang–Mills theory (or $D = 4$, $N = 4$)
- $N = (2,0)$ model in $D = 6$
- IIA and IIB supergravities in $D = 10$
- $D = 11$ supergravity
- their dimensional reductions
- BLG model in $D = 3$

How does one formulate an action principle preserving manifest supersymmetry?

Pure spinors provide an answer (in the cases self-dual fields are not present).

Ideas go back to [Nilsson 1986, Howe 1991,...]. Breakthrough in [Berkovits 2000].

Torsion in flat superspace, generically:

$$\{D_\alpha, D_\beta\} = -T_{\alpha\beta}{}^c D_c = -2\gamma^c_{\alpha\beta} D_c \ .$$

If a bosonic spinor λ^α is *pure*, *i.e.*, if the vector part $(\lambda\gamma^a\lambda)$ of the spinor bilinear vanishes, the operator $q = \lambda^\alpha D_\alpha$ becomes nilpotent,

$$q^2 = 0 \ ,$$

and may be used as a BRST operator. Physical states may be defined as cohomology of q.

Comments:

Solution of the pure spinor constraint typically demands a complex spinor.

λ^α is a set of ghost variables for a first-quantised superparticle.

Physical states are defined as those annihilated by q,

$$q\Psi = 0 \ ,$$

modulo gauge transformations

$$\delta_\Lambda \Psi = q\Lambda \ .$$

where $\Psi = \Psi(x, \theta, \lambda)$.

Input:

dimensionality and spinor module

purity constraint on λ

representation of field Ψ

Cohomology

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Output:

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I will start with the canonical example, $D = 10$ super-Yang–Mills theory.

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I will start with the canonical example, $D = 10$ super-Yang–Mills theory.

Chiral λ^α (16 components)
$(\lambda\gamma^a\lambda) = 0$
Scalar Ψ with gh# 1.

The solution of the pure spinor constraint $(\lambda\gamma^a\lambda) = 0$ leaves us with λ describing a cône of complex dimension 11. The surviving spinor bilinear is $(\lambda\gamma^{abcde}\lambda)$.

First, calculate the cohomology of zero-modes, *i.e.*, with the restriction $\partial_a\Psi = 0$. Note that this is a purely algebraic calculation.

	gh# = 1	0	−1	−2	−3
dim = 0	(00000)				
$\frac{1}{2}$	•	•			
1	•	(10000)	•		
$\frac{3}{2}$	•	(00001)	•	•	
2	•	•	•	•	•
$\frac{5}{2}$	•	•	(00010)	•	•
3	•	•	(10000)	•	•
$\frac{7}{2}$	•	•	•	•	•
4	•	•	•	(00000)	•
$\frac{9}{2}$	•	•	•	•	•

Table 1. The zero-mode cohomology of the $D = 10$ SYM complex.

[Cederwall, Nilsson, Tsimpis 2001; Berkovits 2001]

		gh# $= 1$	0	-1	-2	-3
	dim $= 0$	(00000)				
	$\frac{1}{2}$	•	•			
A_a	1	•	(10000)	•		
χ^α	$\frac{3}{2}$	•	(00001)	•	•	
	2	•	•	•	•	•
	$\frac{5}{2}$	•	•	(00010)	•	•
	3	•	•	(10000)	•	•
	$\frac{7}{2}$	•	•	•	•	•
	4	•	•	•	(00000)	•
	$\frac{9}{2}$	•	•	•	•	•

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		gh# = 1	0	−1	−2	−3
c	dim = 0	(00000)				
	$\frac{1}{2}$	•	•			
A_a	1	•	(10000)	•		
χ^α	$\frac{3}{2}$	•	(00001)	•	•	
	2	•	•	•	•	•
	$\frac{5}{2}$	•	•	(00010)	•	•
	3	•	•	(10000)	•	•
	$\frac{7}{2}$	•	•	•	•	•
	4	•	•	•	(00000)	•
	$\frac{9}{2}$	•	•	•	•	•

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	$\frac{1}{2}$	•	•			
A_a	1	•	(10000)	•		
χ^α	$\frac{3}{2}$	•	(00001)	•	•	
	2	•	•	•	•	•
$\chi^\star_\alpha$	$\frac{5}{2}$	•	•	(00010)	•	•
$A^{\star a}$	3	•	•	(10000)	•	•
	$\frac{7}{2}$	•	•	•	•	•
	4	•	•	•	(00000)	•
	$\frac{9}{2}$	•	•	•	•	•

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	$\frac{1}{2}$	•	•			
A_a	1	•	(10000)	•		
χ^α	$\frac{3}{2}$	•	(00001)	•	•	
	2	•	•	•	•	•
$\chi^\star_\alpha$	$\frac{5}{2}$	•	•	(00010)	•	•
$A^{\star a}$	3	•	•	(10000)	•	•
	$\frac{7}{2}$	•	•	•	•	•
$c^\star$	4	•	•	•	(00000)	•
	$\frac{9}{2}$	•	•	•	•	•

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[Cederwall, Nilsson, Tsimpis 2001; Berkovits 2001]

The zero-mode cohomology provides all the fields and antifields of super-Yang–Mills.

The full cohomology will contain fields given by the modules in the zero-mode cohomology, restricted by differential equations in the modules of the following cohomology.

E.g., the physical fields are subject to (linearised) equations of motion, and similarly for other fields. All comes from $q\Psi = 0$.

The components field at ghost number 0, $\Psi = \lambda^\alpha A_\alpha$, is the fermionic leg of a superspace connection.

In the standard superspace treatment, one demands $F_{\alpha\beta} = 0$. This flatness condition contains two irreducible modules, $(\gamma^a)^{\alpha\beta} F_{\alpha\beta} = 0$ and $(\gamma^{abcde})^{\alpha\beta} F_{\alpha\beta} = 0$. The first of these is a so-called conventional constraint, effectively identifying A_a as sitting in the θ-expansion of A_α. The second is the constraint obtained from $q\Psi = 0$. It enforces the equations of motion.

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Philosophy: if one can find an action whose *equation of motion* is $q\Psi = 0$, it will provide an off-shell superfield action.

It will typically be of the form "$S = \int \Psi q \Psi$".

Non-minimal variables and integration

The cohomology of the ghost antifield at $\lambda^3\theta^5$ has the correct dimension and ghost number to work as a "residue measure". It is however degenerate.

A non-degenerate measure can be formed by introducing non-minimal variables $\bar{\lambda}_\alpha$, r_α, with $(\bar{\lambda}\gamma^a\bar{\lambda}) = 0$, $(\bar{\lambda}\gamma^a r) = 0$.

The non-minimal BRST operator, $Q = \lambda^\alpha D_\alpha + \frac{\partial}{\partial\bar{\lambda}_\alpha} r_\alpha$, has the same cohomology as $q = \lambda^\alpha D_\alpha$.

Integration is full integration over all bosonic and fermionic variables, regulated by $N = \exp(-\{Q, \chi\}) = \exp(-\lambda^\alpha\bar{\lambda}_\alpha - r_\alpha\theta^\alpha)$.

For representatives of cohomology depending on x, λ and θ only, the integration reproduces the residue.

[Berkovits 2005]

A linearised action is

$$S = \int [dZ]\Psi Q \Psi \ .$$

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In order to introduce interaction, the concept of cohomology (which is inherently linear) must be generalised. The appropriate language is the Batalin–Vilkovisky formalism. This is already hinted at by the fact that ghosts and antifields are included in the cohomology.

The action itself is the generator of "gauge transformations", generated as $\delta X = (S, X)$, where $(\cdot,\cdot)$ is the antibracket. In a component formalism:

$$(A, B) = \int [dx] \left(A \frac{\overleftarrow{\delta}}{\delta\phi^A(x)} \frac{\overrightarrow{\delta}}{\delta\phi^\star_A(x)} B - A \frac{\overleftarrow{\delta}}{\delta\phi^\star_A(x)} \frac{\overrightarrow{\delta}}{\delta\phi^A(x)} B \right) .$$

The governing equation generalising $Q^2 = 0$ is the BV master equation $(S, S) = 0$.

[Batalin, Vilkovisky 1981; Zinn-Justin 1975]

For the pure spinor superfield Ψ, the antibracket takes the simple form

$$(A, B) = \int A \frac{\overleftarrow{\delta}}{\delta\Psi(Z)} [dZ] \frac{\overrightarrow{\delta}}{\delta\Psi(Z)} B .$$

[Cederwall 2009]

The full BV action for $D = 10$ super-Yang–Mills (and its dimensional reductions) is the Chern–Simons-like action

$$S = \int [dZ]\mathrm{Tr}\,\left(\tfrac{1}{2}\Psi Q\Psi + \tfrac{1}{3}\Psi^3\right) \ .$$

[Berkovits 2001]

Note that there is only a 3-point coupling; the quartic interaction arises on elimination of "auxiliary fields".

An analogous formulation exists for the Bagger–Lambert–Gustavsson and Aharony–Bergman–Jafferis–Maldacena models in $D = 3$.

The simplification there is even more radical: The component actions contain 6-point couplings, but the pure spinor superfield actions only have minimal coupling (*i.e.*, 3-point interactions).

[Cederwall, 2008]

But I would like to turn to supergravity.

There are two ways of obtaining the linearised on-shell supergravity multiplet from pure spinor superfields. They are connected to the 3-form field and vielbein (on superspace), respectively.

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The pure spinors one has to use fulfill $(\lambda\gamma^a\lambda) = 0$, leaving *two* irreducible modules in the pure spinor bilinear, $(\lambda\gamma^{ab}\lambda)$ and $(\lambda\gamma^{abcde}\lambda)$.

The space of $D = 11$ pure spinors is 23-dimensional. It contains a 16-dimensional singular subspace where $(\lambda\gamma^{ab}\lambda) = 0$, which is the space of $D = 12$ pure spinors.

[Berkovits, Nekrasov 2005; Cederwall 2009; Movshev 2011]

Much of the issues of non-minimal pure spinors and integration goes along roughly the same lines as in $D = 10$, although the presence of reducible modules requires some extra care.

[Berkovits 2002; Anguelova, Grassi, Vanhove 2004; Cederwall 2009]

Quite analogously to SYM, the lowest-dimensional component of the superspace 3-form, $C_{\alpha\beta\gamma}$, can be found in a scalar field Ψ of dimension -3 and gh# 3, whose first component is the 3rd order ghost for the tensor field.

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Similarly, the lowest-dimensional component of the (linearised) super-vielbein, $h_\alpha{}^a$, resides in a vector field Φ^a of dimension -1 and gh# 1, starting out with the diffeomorphism ghost. In addition to the pure spinor constraint, this field has an extra gauge symmetry $\Phi^a \approx \Phi^a + (\lambda\gamma^a\rho)$.

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Each of the corresponding cohomologies contains the full on-shell linearised supermultiplet.

[Cederwall, Nilsson, Tsimpis 2002, Berkovits 2002]

gh# =	3	2	1	0	−1	−2	−3	−4	−5
dim = −3	(00000)								
$-\frac{5}{2}$	•	•							
−2	•	(10000)	•						
$-\frac{3}{2}$	•	•	•	•					
−1	•	•	(01000) (10000)	•	•				
$-\frac{1}{2}$	•	•	(00001)	•	•	•			
0	•	•	•	(00000) (00100) (20000)	•	•	•		
$\frac{1}{2}$	•	•	•	(00001) (10001)	•	•	•	•	
1	•	•	•	•	•	•	•	•	•
$\frac{3}{2}$	•	•	•	•	(00001) (10001)	•	•	•	•
2	•	•	•	•	(00000) (00100) (20000)	•	•	•	•
$\frac{5}{2}$	•	•	•	•	•	(00001)	•	•	•
3	•	•	•	•	•	(01000) (10000)	•	•	•
$\frac{7}{2}$	•	•	•	•	•	•	•	•	•
4	•	•	•	•	•	•	(10000)	•	•
$\frac{9}{2}$	•	•	•	•	•	•	•	•	•
5	•	•	•	•	•	•	•	(00000)	•

Table 2. The cohomology in Ψ.

Note the field-antifield pairing and the existence of a "measure".

Table 3. The cohomology in Ψ*.*

gh# =	3	2	1	0	−1	−2	−3
dim = −3	(00000)						
$-\frac{5}{2}$	•	•					
−2	•	(10000)	•				
$-\frac{3}{2}$	•	•	•	•			
−1	•	•	(01000) (10000)	•	•		
$-\frac{1}{2}$	•	•	(00001)	•	•	•	
0	•	•	•	(00000) (00100) (20000)	•	•	•
$\frac{1}{2}$	•	•	•	(00001) (10001)	•	•	•

Table 4. The cohomology in Φ^a.

gh# =	1	0	−1	−2	−3
dim = −1	(10000)				
$-\frac{1}{2}$	(00001)	•			
0	•	(20000)	•		
$\frac{1}{2}$	•	(00001) (10001)	•	•	
1	•	(00010) (10000)	•	•	•
$\frac{3}{2}$	•	•	(00001) (10001)	•	•
2	•	•	(00000)(00002) (00100)(01000) (10000)(20000)	•	•
$\frac{5}{2}$	•	•	•	•	•
3				(00000)(00002)	

Since both fields describe $D = 11$ supergravity, there should be a relation. Ψ should be the basic field, since it contains the naked tensor potential, without which one can not write the CS term $\int C \wedge H \wedge H$. It also contains ghosts for tensor gauge transformations.

There must be a relation

$$\Phi^a = R^a \Psi$$

where R^a is some operator.

This operator has been constructed:

$$
\begin{aligned}
R^a &= R_0^a + R_1^a + R_2^a \\
&= \eta^{-1}(\bar\lambda\gamma^{ab}\bar\lambda)\partial_b + \eta^{-2}(\bar\lambda\gamma^{ab}\bar\lambda)(\bar\lambda\gamma^{cd}r)(\lambda\gamma_{bcd}D) \\
&- 16\eta^{-3}(\bar\lambda\gamma^{a[b}\bar\lambda)(\bar\lambda\gamma^{cd}r)(\bar\lambda\gamma^{e]f}r)(\lambda\gamma_{fb}\lambda)(\lambda\gamma_{cde}w)\ .
\end{aligned}
$$

It satisfies $[Q, R^a] \approx 0$ (modulo gauge symmetries), and maps cohomology to cohomology.
η is the invariant $(\lambda\gamma^{ab}\lambda)(\bar\lambda\gamma_{ab}\bar\lambda)$, which vanishes on a subspace of complex codimension 7.

(w is the derivative w.r.t. λ. The last term, R_2^a, respects the pure spinor constraint, although not manifestly in the form written here.)

A linearised action (around flat space) is

$$S = \int [dZ] \Psi Q \Psi$$

[Berkovits 2002]

(there is no corresponding action for Φ^a).

This action satisfies the master equation $(S, S) = 0$.

The algebraic properties of the operator R^a ensure that an interaction term

$$S_3 \sim \int [dZ](\lambda\gamma_{ab}\lambda)\Psi R^a \Psi R^b \Psi$$

is a nontrivial deformation respecting the master equation.

The factor $(\lambda\gamma_{ab}\lambda)$ ensures that dimension and gh# are correct, guarantees the invariance under $\Phi^a \approx \Phi^a + (\lambda\gamma^a\rho)$, and makes possible a contraction of Φ^a's.

Some component interactions have been checked explicitly (CS term, coupling of diffeomorphism ghosts), so it is clear that this gives the 3-point couplings of $D = 11$ supergravity.

One may expect that an expansion around flat space would be non-polynomial. This is however not the case. Checking the master equation to higher order in the field involves commutators of R^a's. The R^a's don't commute, but almost:

$$\tfrac{1}{2}(\lambda\gamma_{ab}\lambda)[R^a,R^b]=\tfrac{3}{2}\{Q,T\}$$

where $T=8\eta^{-3}(\bar{\lambda}\gamma^{ab}\bar{\lambda})(\bar{\lambda}r)(rr)(\lambda\gamma_{ab}w)$ is a nilpotent operator of ghost# -3 and dimension 3 (so that $T\Psi$ has gh# 0 and dim. 0). The master equation is *exactly* satified by

$$S=\int[dZ]\left[\tfrac{1}{2}\Psi Q\Psi+\tfrac{1}{6}(\lambda\gamma_{ab}\lambda)(1-\tfrac{3}{2}T\Psi)\Psi R^a\Psi R^b\Psi\right]\ .$$

$$S = \int [dZ] \left[\tfrac{1}{2}\Psi Q \Psi + \tfrac{1}{6}(\lambda\gamma_{ab}\lambda)(1 - \tfrac{3}{2}T\Psi)\Psi R^a \Psi R^b \Psi \right] \ .$$

After a field redefinition $\Psi = (1 + \frac{1}{2}T\tilde{\Psi})\tilde{\Psi}$ the action becomes

$$\begin{aligned} S &= \int [dZ] \left[\tfrac{1}{2}(1 + T\tilde{\Psi})\tilde{\Psi} Q \tilde{\Psi} + \tfrac{1}{6}(\lambda\gamma_{ab}\lambda)\tilde{\Psi} R^a \tilde{\Psi} R^b \tilde{\Psi} \right] \\ &= \int [dZ] \left[\tfrac{1}{2}e^{T\tilde{\Psi}}\tilde{\Psi} Q \tilde{\Psi} + \tfrac{1}{6}(\lambda\gamma_{ab}\lambda)\tilde{\Psi} R^a \tilde{\Psi} R^b \tilde{\Psi} \right] \ . \end{aligned}$$

One has to check that the integration over the pure spinor is well defined, since the operator R^a involves $\eta^{-1} = ((\lambda\gamma^{ab}\lambda)(\bar{\lambda}\gamma_{ab}\bar{\lambda}))^{-1}$.

Gauge fixing

In order to perform quantum calculations based on this formalism, one needs to perform gauge fixing.

Gauge fixing in the BV formalism amount to expressing the anti-fields in terms of the fields, using a “gauge fixing fermion”.

$$\phi*_A = \frac{\partial\chi}{\partial\phi^A} \ .$$

In order to include anti-ghosts and Nakanishi–Lautrup fields, one needs to start from a non-minimal set of fields and antifields.

Gauge fixing of pure spinor superfields relies on an operator b, with $\{b,Q\}=\Box$. This is the "b ghost", which is composite.

[Berkovits 2005]

This operator has been constructed also in $D=11$.

Aisaka, Berkovits, Cederwall, unpublished

$$b=\tfrac{1}{2}\eta^{-1}(\bar\lambda\gamma_{ab}\bar\lambda)(\lambda\gamma^{ab}\gamma^{i}D)\partial_i+\ldots$$

Conclusions

The framework described resolves the issue of supersymmetric actions for maximally supersymmetric theories, in particular $D = 11$ supergravity and its dimensional reductions.

The interaction terms are generically much simpler and of lower order than in a component language; for supergravity to the extent that it becomes polynomial.

Presumably, the formalism is suitable for calculating quantum amplitudes. I would like to examine UV properties of $N = 8$ supergravity. In the present formalism, further regularisation at $\eta = 0$ is probably needed in path integrals... some hope that this will become more tractable in non-covariant ("single-patch") framework...

Lots of other issues. How is U-duality realised? Background invariance? The polynomial property should be better understood.