Double Field Theory of Ramond-Ramond Fields

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1 Joint work with C. Hull, Olaf Hohm and Seung-ki Kwak

1. Introduction

On a torus T^d closed string excitations have momenta p_i and windings w^i $(i = 1, 2, \ldots, d)$.

These are additively conserved quantum numbers and thus have associated coordinates,

$$p_i \leftrightarrow x^i, \qquad w^i \leftrightarrow \tilde{x}_i$$

We say that the x^i coordinates are doubled by the addition of the $\tilde{x}_i$.

All closed string fields are doubled, *i.e.* they have doubled coordinate dependence

$$\phi(x^\mu, \{x^i, \tilde{x}_i\})$$

We will double all coordinates; if there are non-compact coordinates, fields are actually independent of the associated doubles.

$$\phi(x^i, \tilde{x}_i)$$

Closed string field theory (CSFT) is a double field theory

Lagrangian density $\mathcal{L}(x, \tilde{x})$, and action:

$$S = \int dx d\tilde{x} \; \mathcal{L}(x, \tilde{x}) \, .$$

This is doubled field theory!

The $\tilde{x}$ dependence is physical. The x and $\tilde{x}$ dependences of the fields are constrained by the level-matching condition of closed string field theory,

$$(L_0 - \overline{L}_0)|\Psi\rangle \; = \; 0 \, .$$

In toroidal compactifications this gives

$$N - \overline{N} \; = \; p_i \, w^i \qquad (1)$$

We will consider "massless" closed string fields: $N = \overline{N} = 1$.

For them, the constraint (1) requires $p_i w^i = 0$, or equivalently,

$$\frac{\partial}{\partial x^k} \frac{\partial}{\partial \tilde{x}_k} \, \{ g_{ij}(x, \tilde{x}), \; b_{ij}(x, \tilde{x}), \; d(x, \tilde{x}) \} \; = \; 0 \, .$$ WEAK CONSTRAINT

All fields $\phi(x,\tilde{x})$ and all gauge parameters must satisfy the duality covariant constraint

$$\frac{\partial}{\partial x^k}\frac{\partial}{\partial \tilde{x}_k}\,\phi = 0\,, \qquad \forall\phi\,. \tag{2}$$

DFT, so defined, was constructed to cubic order in fluctuations around a flat toroidal background by Hull and BZ. Quartic terms are very complicated.

If we impose the additional constraint that all products of fields satisfy (2):

$$\frac{\partial}{\partial x^k}\frac{\partial}{\partial \tilde{x}_k}\,(\phi_i\phi_j) = 0\,, \tag{3}$$

With this STRONG CONSTRAINT, a full, background independent construction is possible.

If the set of all fields and gauge parameters satisfy the strong constraint then there exists a duality frame where the fields are not doubled:

$$\begin{pmatrix} \tilde{x}' \\ x' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tilde{x} \\ x \end{pmatrix} ,$$

where

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(D,D) \quad \leftrightarrow \qquad h^t \eta h = \eta \, , \qquad \eta \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

In the new duality frame $(\tilde{x}', x')$ the fields are not doubled

$$\phi_k(x, \tilde{x}) = \phi'_k(x')$$

The constraint (both weak or strong) is $O(D,D)$ invariant and the dual frame $(\tilde{x}', x')$ need not be found explicitly.

Thus even the strongly constrained theory will display the $O(D,D)$ symmetry of a doubled field theory, but it is not physically doubled.

(A) DFT with Weak Constraint

Exotic, physically doubled theory. Incomplete construction (do the massless fields suffice?).

(B) DFT with Strong Constraint

T-duality "covariantization" of the conventional theory. Exhibits $O(D, D)$, first step in the construction of (A)

Surprises

The theory in (B) may not be physically the same as the conventional theory.

Allows geometrical interpretation of "non-geometrical" compactifications and fluxes.

Minor relaxations of the strong constraint may be consistent for certain backgrounds.

Explanation for symmetries of dimensionally reduced theories.

In what sense is

$$S_* = \int dx\,\sqrt{-g}\,e^{-2\phi}\Big(R + 4(\partial\phi)^2 - \frac{1}{12}H^2\Big)$$

duality invariant ?

Answer: Upon dimensional reduction on a torus T^d with fields independent of x^i $(i = 1, \ldots, d)$ we get a theory with global $O(d,d)$ symmetry! This is a surprise.

The DFT idea: Double all coordinates and write

$$S_{DFT} \;=\; \int dx d\tilde{x}\,\mathcal{L}(x,\tilde{x})$$

to exhibit manifestly an $O(D,D)$ invariance that would give rise to the suitable $O(d,d)$ upon dimensional reduction.

Assume $\tilde{x} = 0$, then $S_{DFT} = S_*$.

Assume $x = 0$, then $S_{DFT} = S_*$ in some dual variables.

2. NS-NS Sector

$O(D,D)$ indices $M, N, \ldots$ with values running over $1, 2, \ldots, 2D$:

$$Q^M = \{\tilde{q}_1, \tilde{q}_2, \ldots \tilde{q}_d\,,\ q^1, q^2, \ldots q^d\}$$

Thus,

$$X^M = \begin{pmatrix} \tilde{x}_i \\ x^i \end{pmatrix}\,, \quad \partial_M = \begin{pmatrix} \tilde{\partial}^i \\ \partial_i \end{pmatrix}$$

Dualities are $O(D,D)$ transformations:

$$X'^M = h^M{}_N\, X^N$$

As a matrix

$$h^M{}_N = \begin{pmatrix} h_i{}^j & h_{ij} \\ h^{ij} & h^i{}_j \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(D,D)$$

$$h^M{}_P\, h^N{}_Q\, \eta_{MN} \;=\; \eta^{PQ}\,, \qquad \eta^{MN} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \eta_{MN}\,.$$

∞ η^{MN}, with $M, N = 1, 2, \ldots 2D$ is an $O(D,D)$ metric.

The background fields g_{ij} and $b_{ij}(= -b_{ji})$ can be combined into a $D \times D$ matrix $\mathcal{E}$:

$$\mathcal{E}_{ij} = g_{ij} + b_{ij} \quad \rightarrow \quad \mathcal{E} = g + b \in M_D \,.$$

The $O(D,D)$ transformation of $\mathcal{E}$ is

$$\mathcal{E}'(X') = (a\mathcal{E} + b)(c\mathcal{E} + d)^{-1} \,.$$

Use the "generalized metric," a $2D \times 2D$ symmetric matrix constructed from g and b

$$\mathcal{H}_{MN} \;\equiv\; \begin{pmatrix} g^{ij} & -g^{ik}b_{kj} \\ b_{ik}g^{kj} & g_{ij} - b_{ik}g^{kl}b_{lj} \end{pmatrix} \;\equiv\; \mathcal{H}$$

It first appeared in the string Hamiltonian for compactification on tori. It then appeared in generalized geometry.

Never clear how to use it to build the spacetime action.

Tempting to view as a metric in the space $(x, \tilde{x})$.

Using the $O(D,D)$ metric η to raise indices,

$$\mathcal{H}^{MN} \equiv \eta^{MP}\eta^{NQ}\mathcal{H}_{PQ} \,,$$

we find the nontrivial relation

$$\mathcal{H}^{MP}\mathcal{H}_{PQ} = \delta^M_Q\,.$$

$\mathcal{H} \equiv \mathcal{H}_{\bullet\bullet}$ is an $O(D,D)$ tensor:

$$\mathcal{H}'(\mathcal{E}') = (h^{-1})^t\,\mathcal{H}(\mathcal{E})\,h^{-1}\,.$$

Spacetime action:

$$S_{DFT} = \int dx d\tilde{x}\, e^{-2d}\,\Big(\,\frac{1}{8}\,\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\,\partial_N\mathcal{H}_{KL}\;-\frac{1}{2}\mathcal{H}^{MN}\partial_N\mathcal{H}^{KL}\,\partial_L\mathcal{H}_{MK}$$
$$-\,2\,\partial_M d\,\partial_N\mathcal{H}^{MN}\;+\;4\mathcal{H}^{MN}\,\partial_M d\,\partial_N d\Big)\,.$$

$O(D,D)$ symmetry is manifest.

Gauge transformations via generalized Lie derivatives

$$\delta_\xi \mathcal{H}^{MN} = \widehat{\mathcal{L}}_\xi \mathcal{H}^{MN}$$
$$= \xi^P \partial_P \mathcal{H}^{MN} + (\underline{\partial^M \xi_P} - \partial_P \xi^M)\mathcal{H}^{PN} + (\underline{\partial^N \xi_P} - \partial_P \xi^N)\mathcal{H}^{MP}$$

The underlined terms are new (note unusual index position) Gauge algebra

$$\big[\, \delta_{\xi_1}\, , \delta_{\xi_2} \big] = - \, \delta_{[\xi_1,\xi_2]_{\rm C}} \, ,$$

with C-bracket $[\cdot\, , \, \cdot]_{\rm C}$

$$\big[\xi_1, \xi_2\big]_{\rm C}^M \;\equiv\; \xi_{[1}^N \partial_N \xi_{2]}^M - \frac{1}{2}\, \xi_{[1}^P \partial^M \xi_{2]P} \, .$$

For ξ^M that are <u>not doubled</u> the C-bracket reduces to the Courant bracket.

There is a T-duality covariant generalization of the Einstein curvature scalar:

$$\begin{aligned}\mathcal{R} \;\equiv\;\; & 4\,\mathcal{H}^{MN}\partial_M\partial_N d - \partial_M\partial_N\mathcal{H}^{MN} \\ & - 4\,\mathcal{H}^{MN}\partial_M d\,\partial_N d + 4\partial_M\mathcal{H}^{MN}\,\partial_N d\;, \\ & + \frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\,\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\,\partial_K\mathcal{H}_{NL}\;.\end{aligned}$$

$\mathcal{R}$ is a generalized scalar:

$$\delta_\xi\mathcal{R} = \xi^M\partial_M\mathcal{R}\,.$$

The action can then be written as

$$S = \int dx\, d\tilde{x}\, e^{-2d}\,\mathcal{R}\;.$$

There is a T-duality covariant generalized Ricci tensor $\mathcal{R}_{MN}$, but there seems to be no generalized Riemann!!

11 Gauge algebra controlled by the Courant bracket, and generalized metric as dynamical variable. Relations to **generalized geometry** (Weinstein, Courant, Hitchin, Gualtieri).

3. Pin(D,D), Spin(D,D), Spin$^+$(D,D), spinors

Consider the vector space $\mathbb{R}^{2D}$ with a metric $\eta(\cdot\,,\cdot)$.

With basis vectors Γ_M we have $\eta_{MN} = \eta(\Gamma_M, \Gamma_N)$.

Clifford algebra Cliff (D,D):

An algebra generated by a unit element $\mathbf{1}$ and basis vectors in $\mathbb{R}^{2D}$ with a product $\cdot$ that satisfies the relation:

$$V \cdot V = \eta(V, V) \cdot \mathbf{1}\,, \qquad \forall\ V \in \mathbb{R}^{2D}$$

This gives,

$$\rightarrow \quad \Gamma_M \Gamma_N + \Gamma_N \cdot \Gamma_M = 2\eta_{MN}\,\mathbf{1}$$

Define the anti-involution $\star$:

$$(V_1 \cdot V_2 \cdot \ldots V_k)^\star \equiv (-1)^k V_k \cdot V_{k-1} \ldots V_1$$

$$\text{Pin}(D,D) \equiv \{S \in \text{Cliff}(D,D) |\ \ S \cdot S^{\star} = \pm \mathbf{1}\,,\ S \cdot V \cdot S^{-1} \in \mathbb{R}^{2D}, \forall\ V \in \mathbb{R}^{2D}\}$$

$$\rho(S)V \ \equiv\ S \cdot V \cdot S^{-1}$$

Note the group homomorphism,

$$\rho : \ \text{Pin}(D,D) \rightarrow \ O(D,D)$$

Since $\rho(-S) = \rho(S)$ ρ is a double cover map

Other important subgroups

$$\text{Spin}(D,D) \equiv \{S \in \text{Cliff}(D,D)^{\text{even}} | \text{plus all other conditions in Pin}\}$$

$$\rho : \ \text{Spin}(D,D) \rightarrow \ SO(D,D)$$

$$\text{Spin}^{+}(D,D) \equiv \{S \cdot S^{\star} = +\mathbf{1} | \text{plus all other conditions in Spin}(D,D)\}$$

$$\rho : \ \text{Spin}^{+}(D,D) \rightarrow \ SO^{+}(D,D)$$

The homomorphism ρ from the Spin groups to the SO groups, more explicitly is

$$S\,\Gamma_M S^{-1} \;=\; \Gamma_N\, h^N{}_M\,, \qquad \rho:\; S \to h\,.$$

The SO(D,D) group has two disconnected components:

SO$^+$(D,D) which contains the identity.

SO$^-$(D,D) , which is a coset of SO$^+$(D,D)

In a basis where the metric is diag$(\mathbf{1}_D, -\mathbf{1}_D)$ a matrix

$$h = \begin{pmatrix} a & b \\ c & c \end{pmatrix}$$

$h \in$ SO$^+$(D,D) if det(a), det$(d) > 0$.

$h \in$ SO$^-$(D,D) if det(a), det$(d) < 0$.

For <u>Lorentzian</u> signature the generalized metric $\mathcal{H}$ is in

$$\mathcal{H} \;\in \text{SO}^-(\text{D,D})$$

There is no continuous global lift from the space of $\mathcal{H} \in$ SO$^-$(D,D) to Spin$^-$(D,D).

Spinors:

$$\Gamma^M = \begin{pmatrix} \Gamma_i \\ \Gamma^i \end{pmatrix} = \sqrt{2} \begin{pmatrix} \psi_i \\ \psi^i \end{pmatrix}$$

The algebra becomes

$$\{\psi_i, \psi^j\} = \delta_i^j \,, \qquad \{\psi_i, \psi_j\} = \{\psi^i, \psi^j\} = 0$$

$\psi^i \to$ creation ops

$\psi_i \to$ destruction ops

2^D Spinor states

$$\begin{aligned} |0\rangle \\ \psi^i|0\rangle \\ \psi^{i_1}\psi^{i_2}|0\rangle \\ \vdots \\ \psi^1\psi^2 \ldots \psi^D|0\rangle \end{aligned}$$

General spinor encodes all differential forms:

$$|\chi\rangle \;=\; \sum_{p=0}^{D} \frac{1}{p!} C_{i_1 i_2, \ldots i_p}(x, \tilde{x}) \;\psi^{i_1}\psi^{i_2} \ldots \psi^{i_D}|0\rangle \,.$$

Even forms contain 2^{D-1} states

Odd forms contain 2^{D-1} states.

These sets are preserved by the action of elements of Spin(D,D).

Images of Pin(D,D) elements,

$$\rho(\psi^i + \psi_i) = -\begin{pmatrix} 1-e_i & -e_i \\ -e_i & 1-e_i \end{pmatrix} \qquad \text{A T-duality about the } i\text{-th circle}$$

$$\rho\Big(\tfrac{1}{\det r}\exp(\psi^i R_i{}^j \psi_j)\Big) = \begin{pmatrix} r & 0 \\ 0 & (r^{-1})^t \end{pmatrix} \qquad r \in GL^+(D) \text{ and } r = \exp(R).$$

$$\rho\big(\exp(-\tfrac{1}{2}b_{ij}\psi^i\psi^j)\big) = \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}, \qquad b^T = -b$$

$$\rho(\psi^i\psi_i - \psi_i\psi^i) = \begin{pmatrix} k_i & 0 \\ 0 & k_i \end{pmatrix}, \qquad k_i = \text{diag}(1,1,\ldots -1(\text{i-th}) \ldots 1,1)$$

Charge conjugation matrix C $\in$Spin(D,D):

$$C\psi_i C^{-1} = \psi^i \,, \qquad C\psi_i C^{-1} \;=\; \psi^i \,.$$

$$C = (\psi^1 - \psi_1)(\psi^2 - \psi_2)\ldots(\psi^D - \psi_D)\,, \quad C^{-1} = (-1)^{D(D-1)/2} C$$

Defining dagger: $V^\dagger = CVC^{-1}$ for vectors

$$(V_1 \cdots V_n)^\dagger = V_n^\dagger \cdots V_1^\dagger \,, \quad \text{for products}$$

$$\rightarrow \quad (\psi_i)^\dagger = \psi^i \,, \quad (\psi^i)^\dagger = \psi_i \,.$$

Interplay with the homomorphism,

$$\rho(S^\dagger) \;=\; (\rho(S))^T \,.$$

4. RR action

Nice work by M. Fukuma, T. Oota, and H. Tanaka, "Comments on T-dualities of Ramond-Ramond Potentials", Prog. Theor. Phys. **103** (2000) 425.

Restrict χ to even or odd forms (IIA or IIB, a priori)

$$\partial\!\!\!/ \equiv \frac{1}{\sqrt{2}}\Gamma^M \partial_M = \psi^i \partial_i + \psi_i \tilde{\partial}^i$$

$$\partial\!\!\!/^2 = \frac{1}{2}\Gamma^M \Gamma^N \partial_M \partial_N \;=\; \frac{1}{2}\eta^{MN}\partial_M \partial_N \;=\; 0\,.$$

So $\partial\!\!\!/$ is like an exterior derivative in double space. For fields that do not depend on $\tilde{x}$ it is like a familiar exterior derivative.

Recall $|\chi\rangle$ encodes the potentials C_i. If there is no $\tilde{x}$ dependence

$$|F\rangle = \partial\!\!\!/ |\chi\rangle \quad \to \quad F = dC$$

RR fields live in χ, a spinor of Spin(D,D).

Fundamental gravitational field

$$\mathbb{S} \in \text{Spin}^{-}(\text{D,D})$$

We do not attempt to lift $\mathcal{H}$ to Spin^{-}(D,D) because of the global problem mentioned earlier.

The generalized metric is recovered from the homomorphism ρ : Spin^{-}(D,D) $\rightarrow$ SO^{-}(D,D)

$$\rho(\mathbb{S}) = \mathcal{H}\,.$$

Since the generalized metric is symmetric we demand

$$\mathbb{S}^{\dagger} = \mathbb{S}\,.$$

A very simple DFT action for RR fields!

$$S = \int dx d\tilde{x} \Big(e^{-2d}\,\mathcal{R}(\mathcal{H}, d) + \frac{1}{4} (\partial\!\!\!/ \chi)^{\dagger}\, \mathbb{S}\; \partial\!\!\!/ \chi \Big)\,.$$

Must have a self-duality contraint:

$$\partial\!\!\!/ \chi = -C^{-1} \mathbb{S}\, \partial\!\!\!/ \chi\,.$$

Action evaluation for $\tilde{x} = 0$

Recall $|F\rangle = \partial\!\!\!/ |\chi\rangle \quad \rightarrow \quad F = dC$ so that

$$\frac{1}{4}(\partial\!\!\!/\chi)^\dagger \, \mathbb{S} \, \partial\!\!\!/\chi \;=\; \frac{1}{4}(\partial\!\!\!/\chi)^\dagger \, e^{\frac{1}{2}b_{ij}\psi_i\psi_j} S_g^{-1} e^{-\frac{1}{2}b_{ij}\psi^i\psi^j} \, \partial\!\!\!/\chi$$

Define a new field strength

$$|\widehat{F}\rangle \;=\; e^{-\frac{1}{2}b_{ij}\psi^i\psi^j}|F\rangle \,, \qquad \widehat{F} \;=\; e^{-b^{(2)}} \wedge dC$$

The Lagrangian density is

$$\mathcal{L} \;=\; \frac{1}{4}\,\langle\widehat{F}|S_g^{-1}|\widehat{F}\rangle \;=\; -\frac{1}{4}\,\sqrt{g}\sum_{p=1}^{D}|\widehat{F}^{(p)}|^2$$

The self-duality constraint gives in $D = 10$

$$\widehat{F}^{(p)} \;=\; (-1)^{\frac{1}{2}p(p+1)} \star \widehat{F}^{(D-p)}$$

 This is the so-called democratic formulation of the RR sector.

5. Symmetries and frames

What are the symmetries of the action S?

$$S = \int dx d\tilde{x}\ \frac{1}{4} (\partial\!\!\!/ \chi)^\dagger\, \mathbb{S}\ \partial\!\!\!/ \chi\ .$$

Duality symmetry:

$$\chi \to S\chi \qquad \to \qquad \partial\!\!\!/ \chi \to S \partial\!\!\!/ \chi$$

$$\mathbb{S} \to (S^{-1})^\dagger\, \mathbb{S}\ S^{-1}$$

Originally S ∈Pin(D,D), but the chirality condition on χ means we can only use Spin(D,D).

In addition, the duality symmetry of the self-duality constraint reduces this to Spin$^+$(D,D). So

$$S \in \text{Spin}^+(\text{D,D})\,.$$

The expected consistency condition holds

$$\mathbb{S} \quad \xrightarrow{S} \quad \mathbb{S}' \quad \text{implies} \quad \mathcal{H} \quad \xrightarrow{\rho(S)} \quad \mathcal{H}'\,. \tag{1}$$

p-form gauge symmetry:

$$\delta_\lambda \chi \;=\; \not{\partial} \lambda \,.$$

Invariance holds by virtue of $\not{\partial}^2 \;=\; 0$.

Diffeomorphisms and Kalb-Ramond symmetry ξ^M

$$\delta_\xi \chi \;=\; \xi^M \partial_M \chi + \frac{1}{2} \partial_M \xi_N \, \Gamma^M \Gamma^N \chi$$

The gravitational field must also transform

$$\delta_\xi (C^{-1}\mathbb{S}) \;=\; \xi^M \partial_M (C^{-1}\mathbb{S}) + \frac{1}{2} \big[\Gamma^{PQ}, \, (C^{-1}\mathbb{S}) \, \big] \partial_P \xi_Q \;,$$

One can show this is consistent (via the homomorphism) with the gauge transformation $\delta_\xi \mathcal{H}^{MN} = \widehat{\mathcal{L}} \mathcal{H}^{MN}$.

Constructing Spin representatives $S_{\mathcal{H}}$ of generalized metrics one finds a subtle transformation under T-dualities

$$(S^{-1})^{\dagger} S_{\mathcal{H}} S^{-1} \;=\; \sigma_{\rho(S)}(\mathcal{H}) S_{\rho(S)\circ\mathcal{H}}$$

σ is a sign factor. It is negative for a timelike T-duality and it is positive for a spacelike T-duality. It is negative for a duality inversion about all coordinates.

Pick a chirality for the spinor and pick a sign for $\mathbb{S}$.

Claim that after making such choices, all type II theories are contained in the DFT action and can be read from different duality frames, or different choices of polarizations in the doubled space.

Assume you get IIA for $\tilde{x} = 0$. Then after T-duality in all directions get IIA* theory. After duality on an odd number of space directions get IIB theory.

If we get the “wrong sign” IIA* for $\tilde{x} = 0$, then we get the IIA theory for $x = 0$ (after T-duality in all directions).

6. DFT outlook

Gives a geometric derivation of symmetries that have no simple or clear origin (the nontrivial part of $O(d,d)$ upon compactification).

Can describe massive IIA theory by relaxing a little bit the strong constraint.

Can derive geometrically the so called "non-geometric" compactifications that give rise to general gauged supergravities. Aldazabal, *et.al.* 1109.0290, Geissbühler, 1109.4280.

May be able to do a calculus of higher derivative T-duality invariant Lagrangians.