Integrability Condition for Singular Gauge Transformations

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At this time, the known analytic solutions of open string field theory include solutions for the tachyon vacuum[1, 2, 3], solutions for marginal deformations [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14], and complex solutions in cubic superstring field theory[15]. The current selection is limited, and obviously we'd like to have more solutions. Recently, using singular gauge transformations[16], we have *almost* found solutions for lumps[18] and multiple branes[17], but these solutions suffer from serious singularities which appear to be difficult to consistently fix. In this talk, I'd like to present a formalism which attempts to explain where these singularities come from, and how they should be resolved. It can be thought of as a variant of the formalism for singular gauge transformations introduced by Ellwood[16]. Right now the idea seems uncomfortably formal to me, but still it has some explanatory power, and I think that, at some level, it must be essentially correct.

1 Integrability Condition

Suppose Φ and Ψ are two solutions. A solution U to the equation

$$(Q+\Psi)U = U\Phi \tag{1.1}$$

will be called a *left gauge transformation* from Ψ to Φ, while a solution $\overline{U}$ to the equation

$$-Q\overline{U} + \overline{U}\Psi = \Phi\overline{U} \tag{1.2}$$

will be called a *right gauge transformation* from Ψ to Φ. Note that a left gauge transformation from Ψ to Φ is the same as a right gauge transformation for Φ to Ψ. A left or right gauge transformation with an inverse is a *gauge transformation* in the traditional sense. Obviously, if Ψ and Φ are related by a gauge transformation, they are physically equivalent.

However, Ψ and Φ can be related by left and right gauge transformations even when they are not gauge equivalent. A trivial example is $U = 0$, but there are other ways to do it. For example, given a tachyon vacuum solution Ψ_{tv} with homotopy operator A, we can construct a left gauge transformation from Ψ to Φ as a product of a left gauge transformation of Ψ to the tachyon vacuum,

$$U_1 = 1 + (\Psi - \Psi_{\text{tv}})A \tag{1.3}$$

and a left gauge transformation from the tachyon vacuum to Φ,

$$U_2 = 1 + A(\Phi - \Psi_{\text{tv}}). \tag{1.4}$$

You can easily check

$$(Q + \Psi)(U_1 U_2) = (U_1 U_2)\Phi. \tag{1.5}$$

The construction of left or right gauge transformations is not unique. Given a left gauge transformation U from Ψ to Φ, the string field λU is also a left gauge transformation from Ψ to Φ assuming

$$Q_\Psi \lambda = 0 \tag{1.6}$$

Likewise if $\overline{U}$ is a right gauge transformation $\overline{U}\lambda$ is also a right gauge transformation. Note that λ does not need to be invertible.

The basic question I want to ask is this: given a solution Ψ, under what conditions can a string field U be regarded as a left gauge transformation to another solution Φ? (I will focus on left gauge transformations; the discussion for right gauge transformations is analogous). To answer this question, I am going to imagine the string field as an operator acting (from the left) on the space of half-string functionals, $\mathcal{D}$. Or, alternatively, as an operator acting (from the right) on the dual space of half-string functionals, $\mathcal{D}^*$. Since we know very little about these linear spaces, my considerations will be formal. I will try to illustrate how the formalism works in examples. Back to our question: U can be regarded as a left gauge transformation if and only if $(Q + \Psi)U$ is equal to U times something, in other words

$$\text{Im}_{\mathcal{D}}[(Q + \Psi)U] \subseteq \text{Im}_{\mathcal{D}}\, U \tag{1.7}$$

I will call this the *strong integrability condition.*

The reason we care about this is because given a solution Ψ and a U satisfying the strong integrability condition, we can construct a new solution by formally writing

$$\Phi = U^{-1}(Q + \Psi)U \tag{1.8}$$

This equation is only really interesting if U is not invertible, so that Φ is a physically different solution from Ψ. But in this case we have to explain what (1.8) means. Even when U is not invertible, we can define the operator

$$U^{-1}:\ \mathrm{Im}_{\mathcal{D}}\,U \to \mathcal{D}/\ker_{\mathcal{D}} U \tag{1.9}$$

Then (1.8) determines a new solution Φ up to a string field in the kernel of U. Often this string field can be fixed by considerations of continuity, which often amounts to the requirement that the solution is not too "sliver-like." For example, this is essentially what determines the phantom term in Schnabl's solution, as the sliver state is needed to cancel a sliver-like contribution from the sum over derivatives of wedge states.

So the main obstruction to using U to construct a new solution is the strong integrability condition. Unfortunately, it is not always obvious when $(Q+\Psi)U$ is proportional to U. So I would like to propose a more concrete way to check this. Let's write

$$U = 1 - X \tag{1.10}$$

and suppose that the limit

$$\lim_{N\to\infty} X^N = X^\infty \tag{1.11}$$

converges, in some (unspecified) sense, to a well defined projector. Now note the following facts:

Fact 1. *A half-string state is annihilated by U if and only if it is proportional to X^∞. In other words,*

$$\ker_{\mathcal{D}} U = \mathrm{Im}_{\mathcal{D}}\, X^\infty, \qquad \ker_{\mathcal{D}^*} U = \mathrm{Im}_{\mathcal{D}^*}\, X^\infty. \tag{1.12}$$

Proof. First show that $\mathrm{Im}_{\mathcal{D}}\, X^\infty \subseteq \ker_{\mathcal{D}} U$. If $|v\rangle \in \mathrm{Im}_{\mathcal{D}}\, X^\infty$, then $|v\rangle = X^\infty|w\rangle$ for some $|w\rangle$. Then $U|v\rangle = (1-X)X^\infty|w\rangle = (X^\infty - X^\infty)|w\rangle = 0$, so $|v\rangle \in \ker_{\mathcal{D}} U$. Next show $\ker_{\mathcal{D}} U \subseteq \mathrm{Im}_{\mathcal{D}}\, X^\infty$. If $|v\rangle \in \ker_{\mathcal{D}} U$, then $|v\rangle = X|v\rangle$. Applying this iteratively implies $|v\rangle = X^N|v\rangle$ and in particular $|v\rangle = X^\infty|v\rangle \in \mathrm{Im}_{\mathcal{D}}\, X^\infty$. The result follows. A similar argument applies from the right acting on $\mathcal{D}^*$. □

Fact 2. *If a half string state is proportional to U, it is also annihilated by X^∞.*

$$\mathrm{Im}_{\mathcal{D}}\, U \subseteq \ker_{\mathcal{D}} X^\infty, \qquad \mathrm{Im}_{\mathcal{D}^*}\, U \subseteq \ker_{\mathcal{D}^*} X^\infty. \tag{1.13}$$

Proof. If $|v\rangle \in \mathrm{Im}_{\mathcal{D}}\, U$, then $|v\rangle = (1-X)|w\rangle$ for some $|w\rangle$. Then $X^\infty|v\rangle = (X^\infty - X^\infty)|w\rangle = 0$ and $|v\rangle \in \ker_{\mathcal{D}} X^\infty$. A similar argument applies on $\mathcal{D}^*$. □

The proof of these statements is formal. In the following I will take them to be "axiomatic."

Fact 2 and (1.7) imply what I call the *weak integrability condition*:

$$\mathrm{Im}_{\mathcal{D}}[(Q+\Psi)U] \subseteq \ker_{\mathcal{D}} X^\infty \tag{1.14}$$

or, equivalently

$$X^\infty(Q+\Psi)U = 0 \tag{1.15}$$

Note that it is much easier to check when a state is killed by X^∞ than it is to check whether it is proportional to U. However, since the kernel of X^∞ is in general larger than the image of U, the weak integrability condition is a necessary, but not sufficient, condition for U to be a left gauge transformation. This is a peculiarity of the fact that we are working in infinite dimensions; if the space of half-string functionals were finite dimensional, the weak integrability condition would be both necessary and sufficient.

The strong (and weak) integrability conditions imply a constraint on the projector X^∞ which has an interesting interpretation. To derive it, subtract $Q(X^\infty U) = 0$ from (1.15):

$$(QX^\infty - X^\infty\Psi)U = 0 \tag{1.16}$$

Thus $QX^\infty - X^\infty\Psi$ is in the right kernel of U. By fact 1 this is the right image of X^∞, so we deduce

$$QX^\infty + \Pi X^\infty - X^\infty\Psi = 0 \tag{1.17}$$

where Π is some ghost number 1 string field. To learn something about Π, multiply this equation by U from the left:

$$UQX^\infty + U\Pi X^\infty = 0 \tag{1.18}$$

Again using $Q(UX^\infty) = 0$ this becomes

$$(QU)X^\infty = U\Pi X^\infty \tag{1.19}$$

Now since we assume that U satisfies the strong integrability condition, we know that $QU = U\Phi - \Psi U$ for some solution Φ. Thus

$$U\Phi X^\infty = U\Pi X^\infty \tag{1.20}$$

Thus Π and Φ must be identical up to terms in the left kernel of U and the right kernel of X^∞

$$\Pi = \Phi + X^\infty M + \Pi' U \tag{1.21}$$

Plugging in, we find:

$$QX^\infty + \Phi X^\infty + X^\infty M X^\infty - X^\infty \Psi = 0 \tag{1.22}$$

I will refer to this as the *BRST invariance of* X^∞. Multiplying this by U from the right, we recover the weak integrability condition.

To motivate my interpretation of this equation, consider a wedge state with boundary conditions deformed by a (nonsingular) marginal current V[13]:

$$e^{-(K+V)} = \sigma_{01}\Omega\,\sigma_{10} \tag{1.23}$$

where σ_{01} is a boundary condition changing operator which shifts from the reference boundary conformal field theory, BCFT_0, to the marginally deformed boundary conformal field theory, BCFT_1, and σ_{10} shifts back. Taking the BRST variation of this equation gives

$$-cVe^{-(K+V)} + e^{-(K+V)}cV = (Q\sigma_{01})\Omega\,\sigma_{10} + \sigma_{01}\Omega(Q\sigma_{10}) \tag{1.24}$$

Thus we can informally identify,

$$cV \sim Q\sigma_{10} \tag{1.25}$$

Now note that cV is a solution to the string field theory equations of motion. This suggests a general interpretation: A solution in open string field theory corresponds, from the worldsheet perspective, to the BRST variation of a boundary condition changing operator.

Then an interpretation of (1.22) immediately presents itself: X^∞ is a projector-like state representing a boundary condition changing operator which shifts from the boundary conformal field theory described by Φ to the boundary conformal field theory described by Ψ. Specifically, X^∞ should be considered analogous to the state

$$X^\infty \sim \sigma_{0\Phi}\Omega\,\sigma_{\Phi\Psi}\Omega\,\sigma_{\Psi 0} \tag{1.26}$$

The boundary condition changing operator $\sigma_{\Phi\Psi}$ in the middle represents the shift from the background of Φ to the background of Ψ; the operators $\sigma_{0\Phi}$ and $\sigma_{\Psi 0}$ at the edges are there only because $\sigma_{\Psi\Phi}$ must be expressed relative to the reference boundary conformal field theory. Now take the BRST variation of this state and compare to (1.22): Φ can be interpreted as the BRST variation of $\sigma_{0\Phi}$, M can be interpreted as the BRST variation of $\sigma_{\Phi\Psi}$, and Ψ is the BRST variation of $\sigma_{\Psi 0}$. With this understanding, we can summarize our results as follows:

Summary. *Given a solution* Ψ *describing some boundary conformal field theory* BCFT_Ψ*, a string field* U *is a left gauge transformation only if its associated projector* X^∞ *corresponds to a boundary condition changing operator shifting between* BCFT_Ψ *and some other boundary conformal field theory,* BCFT_Φ*, in the sense of (1.22).*

This is meant only to be a suggestive interpretation, but in some examples the structure of boundary condition changing operators inside X^∞ can be made explicit.

Let me return to the question of how we construct a solution. Given a left gauge transformation U, we can define an inverse operator U^{-1} which maps the image of U to the space of half-string functionals $\mathcal{D}$ modulo the kernel of U, as described in (1.9). Suppose we fix the ambiguity in the kernel of U in some convenient way, defining an operator,

$$(U^{-1})' : \text{Im}_{\mathcal{D}}\, U \to \mathcal{D} \tag{1.27}$$

Then we can write the formal equation (1.8) in a more explicit form:

$$\Phi = (U^{-1})'(Q + \Psi)U + X^\infty \Phi' \tag{1.28}$$

The ghost number 1 string field $X^\infty\Phi'$ reflects the left over ambiguity in the kernel of U. It must be chosen to ensure that the solution is regular and satisfies the equations of motion. In fact, I claim that $X^\infty\Phi'$ is precisely what we mean by the *phantom term* in a general setting.

2 Examples

2.1 Trivial case: $U = 0$

The string field $U = 0$ is a left gauge transformation between any two solutions. The associated projector X^∞ is the identity string field:

$$X^\infty = 1 \tag{2.1}$$

We can interpret this as the projector we get when all of the boundary condition changing operators inside X^∞ have collapsed on top of one another and canceled out. Both the strong and weak integrability conditions are satisfied. Using (1.28) we can therefore express Φ as a formal gauge transformation of Ψ:

$$\Phi = (0^{-1})'(Q + \Psi)0 + \Phi' \tag{2.2}$$

Since the kernel of $U = 0$ is the whole space, we can define the operator $(0^{-1})'$ however we want; in the end it doesn't matter, since in (2.2) it ends up multiplied by zero. Then $\Phi = \Phi'$, and the entire solution consists of the phantom term. Apparently, $U = 0$ does not give us any information about how to construct Φ from Ψ, but with such a trivial equation $(Q + \Psi)0 = 0\Phi$ this is expected.

2.2 Schnabl's solution

As discovered by Okawa[19], Schnabl's solution for the tachyon vacuum can be constructed by a left gauge transformation of the perturbative vacuum $\Psi = 0$:

$$U = 1 - \sqrt{\Omega}cB\sqrt{\Omega} \tag{2.3}$$

The associated projector is,

$$X^{\infty} = \sqrt{\Omega}cB\Omega^{\infty} \tag{2.4}$$

At this point we run into a problem, since B annihilates the sliver in the Fock space. But if we assume $X^{\infty} = 0$, our formalism implies that U is invertible and Schnabl's solution should be pure gauge, which is clearly incorrect for the purposes of string field theory. Therefore, we I will assume

$$B\Omega^{\infty} \neq 0 \tag{2.5}$$

On the other hand, if the limit of X^N for $N \to \infty$ exists, we must have $X^{\infty}U = 0$:

$$X^{\infty}U = \sqrt{\Omega}cB\Omega^{\infty}(1 - \Omega) = 0 \tag{2.6}$$

Since B doesn't kill the sliver, this implies that $1 - \Omega$ kills the sliver, which is equivalent to assuming

$$K\Omega^{\infty} = 0 \tag{2.7}$$

This might appear odd, since in the Fock space both (2.5) and (2.7) vanish in the same way. However, there are reasons why B is different from K in this respect. Note that in the ghost number 0 sector B is always accompanied by a c, whose negative conformal dimension can lead to nonvanishing correlators even when B hits the sliver. Consider for example the identity

$$\Omega^{\infty}(Bc + cB)\Omega^{\infty} = \Omega^{\infty} \tag{2.8}$$

and note on the left hand side B is always multiplying the sliver. A similar mechanism explains why the phantom term in Schnabl's solution is nontrivial, even though it vanishes in the Fock space. By contrast, any finite correlator involving the sliver state will vanish with an extra insertion of K, since the K insertion will compute the derivative of the constant function. If we are allowed to consider correlators where the sliver state diverges (for example $\mathrm{Tr}[c\partial c\Omega^\infty c\Omega^\infty]$), the projector X^∞ most likely does not make sense, which undermines the basic assumption of the formalism. With this motivation, I will simply take (2.5) and (2.7) as formal assumptions. I would still like to give these equations a more concrete and rigorous basis.

According to our interpretation, $X^\infty = \sqrt{\Omega}cB\Omega^\infty$ should represent a boundary condition changing operator between the tachyon vacuum and the perturbative vacuum. On the worldsheet, naively such an operator would be zero. In fact, this might partially explain why X^∞ vanishes in the Fock space; for other solutions, X^∞ will not vanish.

Okawa's gauge transformation should satisfy the weak integrability condition, and, consistently, it does, as is easy to check:

$$\begin{aligned} X^\infty QU &= \sqrt{\Omega}cB\Omega^\infty(cKBc\sqrt{\Omega}) \\ &= \sqrt{\Omega}cB(K\Omega^\infty)c\sqrt{\Omega} \\ &= 0 \end{aligned} \tag{2.9}$$

from (2.7). Anticipating the form of Schnabl's solution, $\Phi = \sqrt{\Omega}c\frac{KB}{1-\Omega}c\sqrt{\Omega}$, and anticipating $M = 0$, we can also check BRST invariance of X^∞:

$$\begin{aligned} QX^\infty + \Phi X^\infty &= -\sqrt{\Omega}cKBc\Omega^\infty + \sqrt{\Omega}c\frac{KB}{1-\Omega}c\Omega cB\Omega^\infty \\ &= -\sqrt{\Omega}cKBc\Omega^\infty + \sqrt{\Omega}c\frac{KB}{1-\Omega}c\Omega\Omega^\infty - \sqrt{\Omega}c\frac{KB}{1-\Omega}\Omega c\Omega^\infty \\ &= -\sqrt{\Omega}cKBc\Omega^\infty + \sqrt{\Omega}c\frac{KB}{1-\Omega}(1-\Omega)c\Omega^\infty \\ &= -\sqrt{\Omega}cKBc\Omega^\infty + \sqrt{\Omega}cKBc\Omega^\infty \\ &= 0 \end{aligned} \tag{2.10}$$

In passing, its worth noting that the phantom term for Schnabl's solution

$$\sqrt{\Omega}cB\Omega^\infty c\sqrt{\Omega} \tag{2.11}$$

is of the form $X^\infty\Phi'$, where Φ' is a ghost number one string field (in this case, $c\sqrt{\Omega}$). This is consistent with (1.28).

We can turn this discussion around and express the perturbative vacuum as a left gauge transformation of Schnabl's solution:

$$U^\ddagger = 1-\sqrt{\Omega}Bc\sqrt{\Omega} \tag{2.12}$$

The associated projector is,

$$(X^\ddagger)^\infty = \Omega^\infty Bc\sqrt{\Omega} \tag{2.13}$$

Let's check the weak integrability condition:

$$\begin{aligned}(X^\ddagger)^\infty(Q+\Phi)U^\ddagger &= -\Omega^\infty Bc\Omega cKBc\sqrt{\Omega}+\Omega^\infty Bc\Omega c\frac{KB}{1-\Omega}c(1-\Omega Bc)\sqrt{\Omega}\\ &= -\Omega^\infty Bc\Omega cKBc\sqrt{\Omega}+\Omega^\infty Bc\Omega c\frac{KB}{1-\Omega}(1-\Omega)c\sqrt{\Omega}\\ &= -\Omega^\infty Bc\Omega cKBc\sqrt{\Omega}+\Omega^\infty Bc\Omega cKBc\sqrt{\Omega}\\ &= 0\end{aligned} \tag{2.14}$$

consistently.

By composing U and $U^\ddagger$, we can transform from the perturbative vacuum to the tachyon vacuum and back. The corresponding left gauge transformation is

$$UU^\ddagger = 1-\sqrt{\Omega}cB\sqrt{\Omega}-\sqrt{\Omega}Bc\sqrt{\Omega} = 1-\Omega \tag{2.15}$$

This state is BRST closed, and if it were invertible, it would be a gauge transformation leaving the perturbative vacuum invariant. However, this state is not invertible, and the associated projector is

$$(X+X^\ddagger)^\infty = \Omega^\infty \tag{2.16}$$

This is a sliver-like representation of the unit operator, which is the "boundary condition changing operator" relating the perturbative vacuum to itself. Note that, unlike (2.4), this projector does not vanish in the Fock space. Also note that the trivial shift in boundary condition is easiest to see by using a left gauge transformation which passes through the tachyon vacuum. A simpler gauge transformation $U=1$ leads to $X^\infty=0$, which does not lend itself to any particular interpretation in terms of boundary condition changing operators.

2.3 Multibranes and Ghost Branes

Now let us consider multibrane and ghost brane solutions[17], which are known to suffer from singularities related to their definition as a formal gauge transformation of the perturbative vacuum. The two-brane solution can be derived by applying the left gauge transformation $U^\ddagger$ in (2.12)—which takes the tachyon vacuum to the perturbative vacuum—once again to the perturbative vacuum. The projector $(X^\ddagger)^\infty = \Omega^\infty Bc\sqrt{\Omega}$ is the same as before, but since we are starting from the perturbative vacuum the weak integrability condition is different:

$$\begin{aligned}(X^\ddagger)^\infty QU^\ddagger &= -\Omega^\infty Bc\Omega cKBc\sqrt{\Omega}\\ &= -\Omega^\infty c(1-\Omega)KBc\end{aligned} \tag{2.17}$$

This state is not zero (from the perspective of our formalism), and so we expect the 2-brane solution to be singular. Indeed it appears to be. This does not mean that the 2-brane solution, as currently defined, cannot be fixed with some prescription; but I think this prescription would ultimately go beyond the current singular pure gauge ansatz.

Ghost brane solutions are defined by applying the left gauge transformation U in (2.3) more than once to the perturbative vacuum. For example, the -2 brane solution is defined by

$$U^2 = 1 - \sqrt{\Omega}cB(2-\Omega)\sqrt{\Omega} \tag{2.18}$$

The associated projector is

$$(2X - X^2)^\infty = \sqrt{\Omega}cB\left[\lim_{N\to\infty}\Omega^{N-1}(2-\Omega)^N\right]\sqrt{\Omega} \tag{2.19}$$

One can check that the limit in the brackets approaches the sliver state. However, it is interesting to note that with respect to the (conjectural) C^* norm on the algebra of wedge states, this does not approach the "same" sliver state as Ω^N for large N. A similar phenomenon was observed in the study of the phantom term in half-brane solutions in cubic superstring field theory[15]. At any rate, the current formalism does not appear to be refined enough to appreciate this distinction, and we have simply

$$(2X - X^2)^\infty = \sqrt{\Omega}cB\Omega^\infty \tag{2.20}$$

This is the same projector as (2.4), and by an identical calculation, the weak integrability condition condition is satisfied:

$$(2X - X^2)^\infty QU^2 = 0 \tag{2.21}$$

This is an example of a left gauge transformation which satisfies the weak integrability condition but which, nevertheless, defines a singular solution. The problem is that the weak integrability condition does not imply the strong integrability condition. The basic point can be appreciated as follows: Consider the vector space of bounded functions on the positive real line $K \geq 0$. On this vector space, consider the operator of multiplication by the function $U(K)$, where $0 \leq U(K) \leq 1$ and $U(K)$ vanishes quadratically at $K = 0$. Defining $U(K) = 1 - X(K)$, the operator

$$X(K)^\infty = \begin{cases} 1 & \text{for} \quad K = 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.22}$$

is the projector onto the kernel of $U(K)$, as described in fact 1. However, the kernel of $X(K)$ consists of all functions which vanish at $K = 0$, whereas the image of $U(K)$ only consists of functions which vanish quadratically. So for example, the equation

$$Ke^{-K} = U(K)f(K) \tag{2.23}$$

has no solution for a bounded function $f(K)$, even though both sides of this equation are annihilated by $X(K)$. This is essentially what's going on for the ghost brane solutions.

2.4 Solutions of Kiermaier, Okawa, and Soler

The solutions considered so far are universal, so its hard to see the relationship between X^∞ and boundary condition changing operators. So let's consider the class of non-universal solutions introduced by Kiermaier, Okawa, and Soler[13] (the KOS solutions):

$$\Phi_1 = -(c\partial\sigma_{01})\frac{1}{1+K}\sigma_{10}(1+K)Bc\frac{1}{1+K} \tag{2.24}$$

This solution describes an open string background BCFT_1 with the property that the boundary condition changing operators σ_{01} and σ_{10}, shifting between BCFT_0 and BCFT_1 and back, are dimension zero primaries[1] satisfying $\sigma_{01}\sigma_{10} = 1$. We can construct a left gauge transformation relating KOS to the simple tachyon vacuum[2]

$$\Psi_{\text{tv}} = (c + Q(Bc))\frac{1}{1+K} \tag{2.25}$$

[1] I assume σ_{01} and σ_{10} are dimension 0 primaries for simplicity. I put the security strip on the right to make shorter formulas.

using the formula (1.3):

$$\begin{aligned}
U_1 &= 1 + (\Phi_1 - \Psi_{\rm tv})A \\
&= 1 + \left[-c\partial\sigma_{01}\frac{1}{1+K}\sigma_{10}(1+K)Bc\frac{1}{1+K} - (c + cKBc)\frac{1}{1+K}\right]\frac{B}{1+K} \\
&= 1 - cB[K+1,\sigma_{01}]\frac{1}{1+K}\sigma_{10}\frac{1}{1+K} - cB\frac{1}{1+K} \\
&= 1 - cB(1+K)\sigma_{01}\frac{1}{1+K}\sigma_{10}\frac{1}{1+K}
\end{aligned} \tag{2.26}$$

This gives

$$X_1 = cB(1+K)\sigma_{01}\frac{1}{1+K}\sigma_{10}\frac{1}{1+K} \tag{2.27}$$

and the projector

$$X_1^\infty = cB(1+K)\sigma_{01}\Omega^\infty\sigma_{10}\frac{1}{1+K} \tag{2.28}$$

On the other hand, we can transform from the tachyon vacuum to a KOS solution describing a different boundary conformal field theory BCFT_2:

$$\begin{aligned}
U_2 &= 1 + A(\Phi_2 - \Psi_{\rm tv}) \\
&= 1 + \frac{B}{1+K}\left[-c\partial\sigma_{02}\frac{1}{1+K}\sigma_{20}(1+K)Bc\frac{1}{1+K} - (c + cKBc)\frac{1}{1+K}\right] \\
&= 1 - \frac{1}{K+1}[K+1,\sigma_{02}]\frac{1}{1+K}\sigma_{20}(1+K)Bc\frac{1}{1+K} - Bc\frac{1}{1+K} \\
&= 1 - \sigma_{02}\frac{1}{1+K}\sigma_{20}(1+K)Bc\frac{1}{1+K}
\end{aligned} \tag{2.29}$$

This gives

$$X_2 = \sigma_{02}\frac{1}{1+K}\sigma_{20}(1+K)Bc\frac{1}{1+K} \tag{2.30}$$

and the projector

$$X_2^\infty = \sigma_{02}\Omega^\infty\sigma_{20}(1+K)Bc\frac{1}{1+K} \tag{2.31}$$

To really see a shift in boundary condition we should construct a left gauge transformation from Φ_1 to Φ_2:

$$U_{12} = U_1U_2 = 1 - \left[cB(1+K)\sigma_{01}\frac{1}{1+K}\sigma_{10} + \sigma_{02}\frac{1}{1+K}\sigma_{20}(1+K)Bc\right]\frac{1}{1+K} \tag{2.32}$$

Without calculating anything, we can already anticipate what the projector X_{12}^∞ should look like: U_{12} has a left kernel corresponding to X_2^∞ and a right kernel corresponding to X_1^∞, so we should expect

$$X_{12}^\infty \sim X_2^\infty X_1^\infty \tag{2.33}$$

We already see the change in boundary condition between BCFT_2 and BCFT_1 at the midpoint. What actually happens is a little more complex, but is essentially captured by this expectation. To calculate X_{12}^∞ I found it convenient to use the formula

$$X_{12}^\infty = \lim_{\epsilon\to 0}\frac{\epsilon}{1-(1-\epsilon)X_{12}} \tag{2.34}$$

Plug in,

$$\begin{aligned}
&\lim_{\epsilon\to 0}\frac{\epsilon}{1-(1-\epsilon)\left[cB(1+K)\sigma_{01}\frac{1}{1+K}\sigma_{10}+\sigma_{02}\frac{1}{1+K}\sigma_{20}(1+K)Bc\right]\frac{1}{1+K}}\\
&=\lim_{\epsilon\to 0}\epsilon\cdot\left(\frac{1}{1-(1-\epsilon)\sigma_{02}\frac{1}{1+K}\sigma_{20}(1+K)Bc\frac{1}{1+K}}\right)\left(\frac{1}{1-(1-\epsilon)cB(1+K)\sigma_{01}\frac{1}{1+K}\sigma_{10}\frac{1}{1+K}}\right)\\
&=\lim_{\epsilon\to 0}\epsilon\cdot\left(1+\sigma_{02}\frac{1-\epsilon}{K+\epsilon}\sigma_{20}(1+K)Bc\frac{1}{1+K}\right)\left(1+cB(1+K)\sigma_{01}\frac{1-\epsilon}{K+\epsilon}\sigma_{10}\frac{1}{1+K}\right)\\
&=\lim_{\epsilon\to 0}\left[\left(\sigma_{02}\frac{\epsilon}{K+\epsilon}\sigma_{20}\right)(1+K)Bc\frac{1}{1+K}+cB(1+K)\left(\sigma_{01}\frac{\epsilon}{K+\epsilon}\sigma_{10}\right)\frac{1}{1+K}\right.\\
&\qquad\left.+\,\epsilon\left(\sigma_{02}\frac{1}{K+\epsilon}B\sigma_{21}\partial c\frac{1}{K+\epsilon}\sigma_{10}\right)\frac{1}{1+K}\right]
\end{aligned}\tag{2.35}$$

In the last step I multiplied out the terms and dropped factors of $1-\epsilon$ which I don't think effect the limit; I also assumed that we could write $\sigma_{20}\sigma_{01}=\sigma_{21}$, where σ_{21} is the boundary condition changing operator between BCFT_2 and BCFT_1. This assumption is not true if σ_{10} and σ_{02} have singular OPE; I'm not sure yet how this difficulty should be interpreted or resolved. I'm also, for the moment, reserving judgment about how the limit $\epsilon\to 0$ should be interpreted, especially for the third term. However, some things are already clear: The first and second terms are the projectors X_2^∞ and X_1^∞ describing boundary condition changing operators between BCFT_2 and BCFT_1 and the tachyon vacuum; from the worldsheet perspective, these terms should be interpreted as zero. The third term, however, clearly can be interpreted as the boundary condition changing operator shifting between BCFT_2 and BCFT_1. One sanity check is to set all of the boundary condition changing operators above equal to unity, which should describe the left gauge transformation from the perturbative vacuum to itself. The limit $\epsilon\to 0$ above can then be computed exactly, and we find the sliver state.

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