Dimensional regularization of Witten's OSFT

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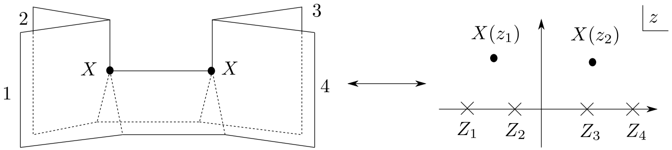
Introduction

Witten's superstring field theory



$$A = \int dt \left\langle X(z_1) X(z_2) \int b \prod_{r=1}^{4} V_r \right\rangle$$

Contact term divergence $z_1 \to z_2$
(Aref'eva and Medvedev, Wendt)

Ways to avoid the problem
(modified cubic, Berkovits, Berkovits and Siegel, democratic, ...)

Dimensional regularization

In the light-cone gauge SFT, we can deal with the problem by dimensional regularization. (Baba, Murakami and N.I.)

- By considering the amplitudes for noninteger d, one can regularize the divergence.
- For tree amplitudes, the limit $d \to 10$ is smooth and the results coincide with those of the first quantized formalism.

We would like to see if dimensional regularization is possible also in the Witten's theory.

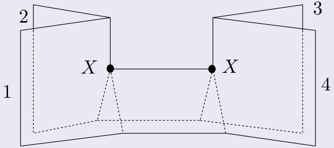
Dimensional regularization for Witten's theory?

Idea

We consider noncritical strings with worldsheet theory

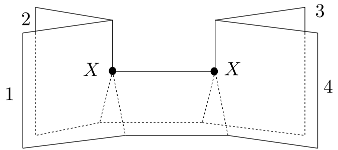
$$X^\mu, \psi^\mu \ (\mu = 0, 1, \cdots, d-1) \oplus \text{ghosts} \oplus \varphi, \cdots .$$

- φ is a variable which works as a metric on the worldsheet.
- φ is taken to reflect the shape of the Feynman diagram.



- We arrange $\varphi, \cdots$ so that in the limit $d \to 10$ we recover the expression of the amplitudes for critical strings.

How $\varphi, \cdots$ regularize the divergence



$$\int dt \left\langle X(z_1) X(z_2) \int b \cdots \prod_{r=1}^{4} V_r \right\rangle_{X^\mu, \text{ghosts}, \varphi, \cdots}$$

Since in the limit $z_1 \to z_2$, the shape of the Feynman diagram becomes singular, we expect $\langle \cdots \rangle$ becomes also singular. If

$$\langle \cdots \rangle \sim |z_1 - z_2|^{-c(d-10)+\text{const}} ,$$

the amplitudes become finite by adjusting d.

Dimensional regularization for Witten's theory

In this talk, I would like to show that such a regularization is possible at least for tree amplitudes. (to appear, with K. Murakami)

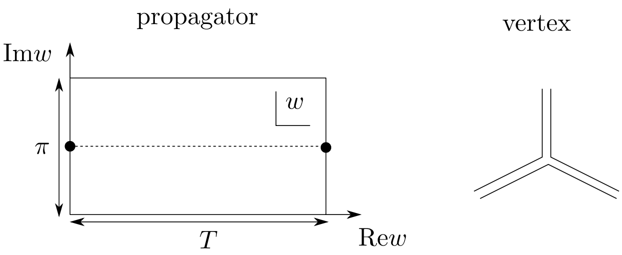
- $\varphi \sim$ the quadratic differential
- It is possible to construct such a noncritical string theory for bosonic strings and write down the amplitudes for $d \neq 26$ in a BRST invariant way.
- Supersymmetrizing the construction, one can deal with the contact term divergence.

Outline

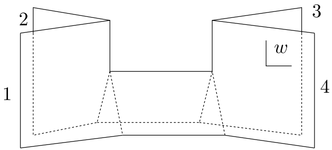
1. Quadratic differential
2. Worldsheet theory for $d = 26$
3. Worldsheet theory for $d \neq 26$
4. Superstrings
5. Conclusions

§1 Quadratic differential

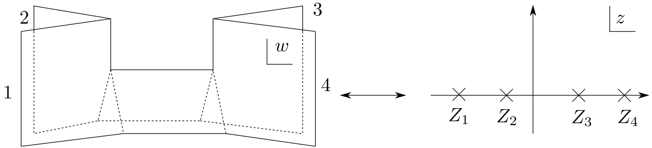
Feynman diagram in the Siegel gauge



One can define the coordinate w on Feynman diagrams



Quadratic differential



quadratic differential

$$
\begin{aligned}
dw^2 &= \phi(z)\, dz^2 , \\
w(z) &= \int^z dz' \sqrt{\phi(z')}
\end{aligned}
$$

metric

$$ds^2 = dw d\bar{w} = \sqrt{\phi(z)\, \bar{\phi}(\bar{z})} dz d\bar{z}$$

ϕ is exactly what we want to have on the worldsheet of the noncritical theory.

The quadratic differential satisfies

- For $z \sim Z_r \ (r = 1, \cdots, N)$

$$\phi(z) \sim \frac{1}{(z - Z_r)^2} + \cdots .$$

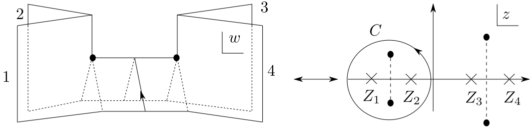
- $z_I \ (I = 1, \cdots, N-2)$: the interaction points

$$\phi(z_I) = 0 .$$

- Since the open strings are of width π

$$\oint_{C_{\mathcal{I}}} \frac{dz}{2\pi i} \sqrt{\phi(z)} = 1 .$$

$\mathcal{I} = 1, \cdots, N-3$: labels for the internal lines back



Quadratic differential on the worldsheet

We would like to realize the quadratic differential as (an expectation value of) a variable on the worldsheet

$$I_{\varphi\varpi} = \frac{1}{\pi}\int d^2 z \left(\varphi\bar{\partial}\varpi + \text{antiholomorphic}\right)$$

$$\begin{aligned} \varphi &: \text{weight } 2 \\ \varpi &: \text{weight } -1 \end{aligned}$$

We insert operators at $z = Z_r, z_I$ etc. so that φ has an expectation value which coincides with the quadratic differential ϕ for the Feynman diagram.

Operator insertions

$$\int [d\varphi d\varpi]\, e^{-I_{\varphi\varpi}} \prod_r \delta(\varpi)\, e^{-\partial\varpi}(Z_r) \prod_{\mathcal{I}} \delta\left(\oint_{C_{\mathcal{I}}} \frac{dz}{2\pi i}\sqrt{\varphi} - 1\right)$$
$$\times \prod_I \left[\lim_{\varepsilon_I\to 0} \oint_{|z-z_I|=\varepsilon_I} \frac{dz}{2\pi i}\partial(\ln\varphi) \oint_{|z-\bar{z}_I|=\varepsilon_I} \frac{dz}{2\pi i}\partial(\ln\varphi)\right]$$

- With $\prod_r \delta(\varpi)\, e^{-\partial\varpi}(Z_r)$, φ has an expectation value of the form $\varphi_{\mathrm{cl}}(z) = \sum_{r=1}^{N}\left(\frac{1}{(z-Z_r)^2} + \frac{a_r}{z-Z_r}\right)$. go
- Other insertions fix a_r so that φ_{cl} coincides with the quadratic differential ϕ. go

Thus we get a theory with φ having an expectation value which coincides with the quadratic differential ϕ.

Quadratic differential on the worldsheet

$$\int [d\varphi d\varpi]\, e^{-I_{\varphi\varpi}} \prod_r \delta(\varpi)\, e^{-\partial\varpi}(Z_r) \prod_{\mathcal{I}} \delta\left(\oint_{C_{\mathcal{I}}} \frac{dz}{2\pi i}\sqrt{\varphi} - 1\right)$$
$$\times \prod_I \left[\lim_{\varepsilon_I \to 0} \oint_{|z-z_I|=\varepsilon_I} \frac{dz}{2\pi i}\partial(\ln\varphi) \oint_{|z-\bar{z}_I|=\varepsilon_I} \frac{dz}{2\pi i}\partial(\ln\varphi)\right]$$

- φ can work as a metric on the worldsheet.

$$ds^2 = \sqrt{\varphi(z)\,\bar{\varphi}(\bar{z})}\,dzd\bar{z}$$

- φ has an expectation value which coincides with ϕ which reflects the shape of the Feynman diagram.
- $\oint \frac{dz}{2\pi i}\sqrt{\varphi}$, , $\oint_{z_I} \frac{dz}{2\pi i}\partial(\ln\varphi)$ commute with $T(z)$. (good for BRST invariance) Rem

This theory can be used in constructing the noncritical string theory.

§2 Worldsheet theory for $d = 26$

We would like to incorporate φ, ϖ into the worldsheet theory of the critical strings.

$$X^\mu \oplus \text{ghosts} \oplus \varphi, \varpi, \hat{b}, \hat{c}$$

We introduce $\hat{b}, \hat{c}$ in order not to spoil the BRST symmetry

$$I_{\varphi\varpi\hat{b}\hat{c}} = \frac{1}{\pi}\int d^2z \left(\varphi\bar{\partial}\varpi + \hat{b}\bar{\partial}\hat{c} + \text{antiholomorhpic}\right) ,$$

$$\left.\begin{array}{ll} \hat{b} & \text{weight } 2 \\ \hat{c} & \text{weight } -1 \end{array}\right\} \text{Grassmann odd}$$

$$c_{tot} = 0$$

String field theory with $\varphi, \varpi, \hat{b}, \hat{c}$

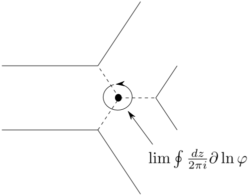
Action

$$\int \left(\frac{1}{2} \Psi \cdot Q_{\mathrm{B}} \Psi + \frac{1}{3} \Psi \cdot \Psi * \Psi \right)$$

- We impose the conditions

$$\left(\int \frac{dz}{2\pi i} \sqrt{\varphi} - 1 \right) \Psi = \int dz \frac{\hat{b}}{\sqrt{\varphi}} \Psi = 0$$

- define the star product with the insertion of $\lim_{\varepsilon_I \to 0} \oint_{|z-z_I|=\varepsilon_I} \frac{dz}{2\pi i} \partial \left(\ln \varphi \right) \oint_{|z-\bar{z}_I|=\varepsilon_I} \frac{dz}{2\pi i} \partial \left(\ln \varphi \right)$



Gauge invariance is OK.

Amplitudes

With such an action, the amplitudes with external vertex operators $\hat{c}\delta\left(\varpi\right)e^{-\partial\varpi}cV_r^X$ are given as

$$
\begin{aligned}
A = & \int\prod_{\mathcal{I}}dt_{\mathcal{I}}\left\langle \prod_{\mathcal{I}}\oint_{C_{\mathcal{I}}}dwb_{ww}\prod_{r}cV_r^X\left(Z_r\right)\right\rangle_{X^\mu,b,c} \\
& \times\left\langle \prod_{\mathcal{I}}\oint_{C_{\mathcal{I}}}dz\frac{\hat{b}}{\sqrt{\varphi}}\delta\left(\oint_{C_{\mathcal{I}}}\frac{dz}{2\pi i}\sqrt{\varphi}-1\right)\right. \\
& \times\prod_{r}\hat{c}\delta\left(\varpi\right)e^{-\partial\varpi}\left(Z_r\right) \\
& \left.\times\prod_{I}\left[\oint_{z_I}\frac{dz}{2\pi i}\partial\left(\ln\varphi\right)\oint_{\bar{z}_I}\frac{dz}{2\pi i}\partial\left(\ln\varphi\right)\right]\right\rangle_{\varphi\varpi\hat{b}\hat{c}} .
\end{aligned}
$$

- φ acquires an expectation value which coincides with the quadratic differential of the Feynman diagram.
- BRST invariant for on-shell V_r^X.

A coincides with the original one.

$\varphi, \varpi, \hat{b}, \hat{c}$ part

$$\left\langle \prod_{\mathcal{I}} \oint_{C_{\mathcal{I}}} dz \frac{\hat{b}}{\sqrt{\varphi}} \delta \left(\oint_{C_{\mathcal{I}}} \frac{dz}{2\pi i} \sqrt{\varphi} - 1 \right) \right.$$
$$\times \prod_r \hat{c}\delta\left(\varpi\right) e^{-\partial\varpi}\left(Z_r\right)$$
$$\left. \times \prod_I \left[\oint_{z_I} \frac{dz}{2\pi i} \partial\left(\ln\varphi\right) \oint_{\bar{z}_I} \frac{dz}{2\pi i} \partial\left(\ln\varphi\right) \right] \right\rangle_{\varphi\varpi\hat{b}\hat{c}} = 1$$

because

- nilpotent fermionic charge $Q \equiv \oint dz \hat{b}\varpi$ s.t. $T_{\varphi\varpi\hat{b}\hat{c}}\left(z\right) = \left\{Q, -\partial\varphi\varpi - 2\varphi\partial\varpi\right\}$.
- the quantities in the correlation function are Q invariant

Adding $\varphi, \varpi, \hat{b}, \hat{c}$ does not change the amplitudes.

§3 Worldsheet theory for $d \neq 26$

With φ, we can define the worldsheet theory preserving the reparametrization invariance:

$$[dX^\mu \cdots]_\varphi = [dX^\mu \cdots]\, e^{-\frac{d-26}{24} S_{\text{Liouville}}[\ln\sqrt{\varphi\bar{\varphi}}]}$$

The φ, ϖ system is now with the action

$$\frac{1}{\pi}\int d^2z \left(\varphi\bar{\partial}\varpi + \tilde{\varphi}\partial\tilde{\varpi}\right) + \frac{d-26}{24} S_{\text{Liouville}}\left[\ln\sqrt{\varphi\bar{\varphi}}\right]$$

$$S_{\text{Liouville}}[\phi] = -\frac{1}{\pi}\int d^2z \partial\phi\bar{\partial}\phi$$

and the energy-momentum tensor

$$-\partial\varphi\varpi - 2\varphi\partial\varpi - \frac{d-26}{12}\left\{\int^z \sqrt{\varphi}, z\right\}$$

$$\{w, z\} = \frac{\partial^3 w}{\partial w} - \frac{3}{2}\left(\frac{\partial^2 w}{\partial w}\right)^2$$

Amplitudes

We can write down the SFT action and the amplitudes with external vertex operators $\hat{c}\delta\left(\varpi\right)e^{-\partial\varpi}cV_r^X$.

$$
\begin{aligned}
A &= \int\prod_{\mathcal{I}}dt_{\mathcal{I}}\left\langle\prod_{\mathcal{I}}\oint_{C_{\mathcal{I}}}dwb_{ww}\prod_r cV_r^X\left(Z_r\right)\right\rangle_{X^\mu,b,c} \\
&\quad\times\left\langle\prod_{\mathcal{I}}\oint_{C_{\mathcal{I}}}dz\frac{\hat{b}}{\sqrt{\varphi}}\delta\left(\oint_{C_{\mathcal{I}}}\frac{dz}{2\pi i}\sqrt{\varphi}-1\right)\right. \\
&\quad\times\prod_r\hat{c}\delta\left(\varpi\right)e^{-\partial\varpi}\left(Z_r\right) \\
&\quad\times\left.\prod_I\left[\oint_{z_I}\frac{dz}{2\pi i}\partial\left(\ln\varphi\right)\oint_{\bar{z}_I}\frac{dz}{2\pi i}\partial\left(\ln\varphi\right)\right]\right\rangle_{\varphi\varpi\hat{b}\hat{c}}.
\end{aligned}
$$

We would like to check

- the BRST invariance of the amplitudes
- the behavior in the limit $z_I \to z_J$

Noncritical dimensions

In order to answer these questions, we need to study the system of $\varphi, \varpi, \hat{b}, \hat{c}$.

- In the light-cone gauge DR, we encountered such a theory for Abelian differentials ($X^{\pm}$ CFT). We were able to study the theory by
    1. simply calculating the correlation functions starting from the path integral expression
    2. describing the theory via free variables which can be obtained by field redefinition
- In the case at hand, the first method seems to be difficult, and we adopt the second method

Free variables

field redefinition Rem

free variables

$$\varphi, \varpi, \hat{b}, \hat{c} \quad \longleftrightarrow \quad \varphi, \varpi', \hat{b}', \hat{c}'$$

$$\begin{aligned} \hat{c}' &= \varphi^{-\frac{\alpha}{2}} \hat{c} \ : \\ \hat{b}' &= \varphi^{\frac{\alpha}{2}} \hat{b} \ : \\ \varpi' &= \varpi - \frac{\alpha}{2} \frac{\hat{b}\hat{c}}{\varphi} + \frac{3}{8} \alpha \frac{\partial \varphi}{\varphi^2} \ , \end{aligned}$$

$$\frac{1}{2}\alpha^2 + \frac{3}{2}\alpha = \frac{d-26}{24}, \ \begin{cases} \hat{c}' & \text{weight } -1-\alpha \\ \hat{b}' & \text{weight } 2+\alpha \end{cases}$$

Free variables

$$\begin{aligned} T_{\varphi\varpi\hat{b}\hat{c}}(z) &= -\partial\varphi\varpi - 2\varphi\partial\varpi - \frac{d-26}{12}\left\{\int^z \sqrt{\varphi}, z\right\} - \partial\hat{b}\hat{c} - 2\hat{b}\partial\hat{c} \\ &= -\partial\varphi\varpi' - 2\varphi\partial\varpi' - \partial\hat{b}'\hat{c}' - 2\hat{b}'\partial\hat{c}' - \alpha\partial\left(\hat{b}'\hat{c}'\right) \end{aligned}$$

- Using the free variables, one can show

	X^μ, b, c	$\varphi, \varpi, \hat{b}, \hat{c}$
central charge	$d-26$	$26-d$

- The field redefinition is ill-defined at the points where φ_{cl} becomes singular or zero

$$\hat{c}' = \varphi^{-\frac{\alpha}{2}}\hat{c} = (\varphi_{\mathrm{cl}} + \delta\varphi)^{-\frac{\alpha}{2}}\hat{c}\cdots$$

In order to express correlation functions of the original variables in terms of the free variables, we need to insert some operators at these points.

Amplitudes in terms of the free variables

With a slight modification of the vertex operator, the amplitudes are given as

$$\int \prod_{\mathcal{I}} dt_{\mathcal{I}} \left\langle \prod_{\mathcal{I}} \oint_{C_{\mathcal{I}}} dw b_{ww} \prod_r c V_r^X (Z_r) \right\rangle_{X,b,c}$$
$$\times \left\langle \prod_{\mathcal{I}} \oint_{C_{\mathcal{I}}} dz \frac{\hat{b}}{\sqrt{\varphi}} \delta \left(\oint_{C_{\mathcal{I}}} \frac{dz}{2\pi i} \sqrt{\varphi} - 1 \right) \right.$$
$$\times \prod_I \left(\mathcal{O}_I \oint_{z_I} \frac{dz}{2\pi i} \partial (\ln \varphi) \oint_{\bar{z}_I} \frac{dz}{2\pi i} \partial (\ln \varphi) \right)$$
$$\left. \times \prod_r e^{(1+\alpha)\sigma'} \delta (\varpi') e^{-\partial \varpi'} c V_r^X (Z_r) \right\rangle_{\varphi,\cdots}$$

$$\mathcal{O}_I = \oint_{z_I} \frac{dz}{2\pi i} \partial (\ln \varphi) \frac{e^{-\frac{\alpha}{2}\sigma'}}{(\partial\varphi)^{\frac{1}{3}\left(\frac{5}{8}\alpha^2 + \frac{3}{4}\alpha\right)}} (z) \times \text{antiholomorphic}$$
$$\partial \sigma' = \hat{c}' \hat{b}'$$

Rem

Amplitudes in terms of the free variables

With a slight modification of the vertex operator, the amplitudes are given as

$$\int \prod_{\mathcal{I}} dt_{\mathcal{I}} \left\langle \prod_{\mathcal{I}} \oint_{C_{\mathcal{I}}} dw b_{ww} \prod_{r} cV_r^X (Z_r) \right\rangle_{X,b,c}$$
$$\times \left\langle \prod_{\mathcal{I}} \oint_{C_{\mathcal{I}}} dz \frac{\hat{b}}{\sqrt{\varphi}} \delta \left(\oint_{C_{\mathcal{I}}} \frac{dz}{2\pi i} \sqrt{\varphi} - 1 \right) \right.$$
$$\times \prod_{I} \left(\mathcal{O}_I \oint_{z_I} \frac{dz}{2\pi i} \partial (\ln \varphi) \oint_{\bar{z}_I} \frac{dz}{2\pi i} \partial (\ln \varphi) \right)$$
$$\left. \times \prod_{r} e^{(1+\alpha)\sigma'} \delta (\varpi') e^{-\partial \varpi'} cV_r^X (Z_r) \right\rangle_{\varphi,\cdots}$$

- BRST invariant if $\triangle \left(V_r^X \right) = \frac{d-2}{24}$.
- For $z_I \to z_J, \langle \cdots \rangle \sim |z_I - z_J|^{-\frac{d-26}{36}}$

We get a noncritical string theory as desired.

§4 Superstrings

It is possible to supersymmetrize all the above.

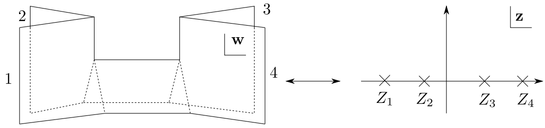
Supersymmetric quadratic differential

$$F(\mathbf{z}) \equiv f(z) + \theta\varphi(z)$$

Superconformal mapping

$$\mathbf{z} = (z, \theta) \to \mathbf{w} = (w, \eta)$$

$$\begin{cases} \eta &= (DF)^{-\frac{3}{4}} F\,, \\ w &= \int^{\mathbf{z}} d\mathbf{z}' \eta D\eta(\mathbf{z}') = \int^{z} dz' \left(\sqrt{\varphi} + \cdots\right) \end{cases}$$



Worldsheet theory

$$X^{\mu}, \psi^{\mu}\ (\mu = 0, \cdots, d-1) \oplus b, c, \beta, \gamma \oplus \varphi, f, \varpi, p, \hat{b}, \hat{c}, \hat{\beta}, \hat{\gamma}$$

$$\begin{aligned} F(\mathbf{z}) &\equiv f(z) + \theta\varphi(z)\ : \text{weight } \frac{3}{2} \\ P(\mathbf{z}) &\equiv \varpi(z) + \theta p(z)\ : \text{weight } -1 \\ \hat{B}(\mathbf{z}) &\equiv \hat{\beta}(z) + \theta\hat{b}(z)\ : \text{weight } \frac{3}{2} \\ \hat{C}(\mathbf{z}) &\equiv \hat{c}(z) + \theta\hat{\gamma}(z)\ : \text{weight } -1 \end{aligned}$$

- Adding these variables does not change the amplitudes for $d = 10$.
- For $d \neq 10$, it is possible to construct free variables $\varphi, f, \varpi', p', \hat{b}', \hat{c}', \hat{\beta}', \hat{\gamma}'$
- Using the free variables, one can show that the total central charge of the super Virasoro algebra vanishes and one can make a nilpotent BRST charge.

Amplitudes in terms of the free variable

We can write down the tree amplitudes (NS sector) in the form

$$A = \int \prod_{\mathcal{I}} dt_{\mathcal{I}} \left\langle \prod_{I=1}^{N-2} \left[X(z_I) \mathcal{O}_I \right] \prod_{\mathcal{I}} \mathcal{P}_{\mathcal{I}} \oint_{C_{\mathcal{I}}} \frac{dw}{2\pi i} b_{ww} \right.$$
$$\left. \times \prod_r \left[c\delta(\gamma)\,\delta(\varpi')\, e^{-\partial\varpi'} p' e^{-(1+\alpha)(\phi'-\sigma')} V_r^X (Z_r) \right] \right\rangle$$

- $\partial\sigma' = \hat{c}'\hat{b}'$, $\partial\phi' = \hat{\gamma}'\hat{\beta}'$ and $\mathcal{O}_I, \mathcal{P}_{\mathcal{I}}$ are complicated combinations of $F, \hat{c}', \hat{b}', \hat{\gamma}', \hat{\beta}'$.
- Everything but $\oint_{C_{\mathcal{I}}} \frac{dw}{2\pi i} b_{ww}$ is BRST invariant if $\triangle\left(V_r^X\right) = \frac{d-2}{16}$.
- In the limit $z_I \to z_J$, $\langle \cdots \rangle \sim |z_I - z_J|^{-\frac{d-10}{24} + \text{const.}}$.

For sufficiently large negative d, there is no contact term divergence.

Dimensional regularization for Witten's theory

For large negative d,

$$\int \prod_{\mathcal{I}} dt_{\mathcal{I}} \left\langle \prod_{I} \left[X\left(z_I\right) \cdots \right] \cdots \prod_{r} V_r^{(p_r)} \right\rangle$$
$$= \int \prod_{\mathcal{I}} dt_{\mathcal{I}} \left\langle \cdots \prod_{r} V_r^{(p'_r)} \right\rangle$$

$d \to 10$ limit is smooth for the expression in the second line and we get the results which coincides those of the first quantized formalism.

§5 Conclusions and discussions

- We define Witten's string field theory in noncritical dimensions.
- Using the noncritical string theory, we can deal with the contact term problem.
- This regularization may be used to study problems in other formulations.
- In order to deal with the multi-loop amplitudes, we need to understand the spectrum of the noncritical string including the closed string sector.

Quadratic differential on the worldsheet

With a choice of the path integral contour for ϖ, $\delta(\varpi)$ can be expressed as $\delta(\varpi(Z_r)) \sim \int da_r e^{-a_r \varpi(Z_r)}$.

$$\int [d\varphi d\varpi] \, e^{-I_{\varphi\varpi}} \prod_{r=1}^{N} \delta(\varpi) \, e^{-\partial\varpi}(Z_r) \cdots$$

$$\sim \int \prod_{r=1}^{N} da_r \int [d\varphi d\varpi] \, e^{-\frac{1}{\pi}\int d^2 z \varphi \bar{\partial}\varpi - \sum_r (a_r \varpi(Z_r) + \partial\varpi(Z_r))} \ldots$$

Solving $\bar{\partial}\varphi_{\mathrm{cl}} = \pi \sum \left(a_r \delta^2(z - Z_r) - \partial\delta^2(z - Z_r)\right)$,

$$\varphi_{\mathrm{cl}}(z) = \sum_{r=1}^{N} \left(\frac{1}{(z - Z_r)^2} + \frac{a_r}{z - Z_r} \right)$$

With the expectation value, we can define $\sqrt{\varphi}$, $\ln\varphi$ etc. almost everywhere. back

$\varphi_{\rm cl}$ coincides with ϕ

$$
\begin{aligned}
\varphi(z,\bar z) &= \varphi_{\rm cl}(z) + \delta\varphi(z,\bar z) , \\
\varpi(z,\bar z) &= \varpi_{-1} z^2 + \varpi_0 z + \varpi_1 + \tilde\varpi(z,\bar z) ,
\end{aligned}
$$

$$
\begin{aligned}
&\int \prod_{r=1}^{N} da_r \int [d\varphi d\varpi]\, e^{-\frac{1}{\pi}\int d^2 z \varphi \bar\partial \varpi - \sum_r (a_r \varpi(Z_r) + \partial\varpi(Z_r))} \ldots \\
&\sim \int \prod_{r=1}^{N} da_r \delta\left(\sum a_r\right) \delta\left(\sum (a_r Z_r + 1)\right) \delta\left(\sum (a_r Z_r^2 + 2 Z_r)\right) \\
&\qquad \times \int [d\delta\varphi d\tilde\varpi]\, e^{-\frac{1}{\pi}\int d^2 z \delta\varphi \bar\partial \tilde\varpi} \ldots
\end{aligned}
$$

- Since $\cdots$ does not involve ϖ, we can replace φ in $\cdots$ by $\varphi_{\rm cl}$.

$\varphi_{\rm cl}$ coincides with ϕ

$$\int \prod_{r=1}^{N} da_r \delta\left(\sum a_r\right)\delta\left(\sum (a_r Z_r+1)\right)\delta\left(\sum\left(a_r Z_r^2+2Z_r\right)\right)$$
$$\times \prod_{\mathcal{I}=1}^{N-3}\delta\left(\oint_{C_{\mathcal{I}}}\frac{dz}{2\pi i}\sqrt{\varphi_{\rm cl}}-1\right)$$
$$\times \prod_{I}\left[\lim_{\varepsilon_I\to 0}\oint_{|z-z_I|=\varepsilon_I}\frac{dz}{2\pi i}\partial\left(\ln\varphi_{\rm cl}\right)\oint_{|z-\bar{z}_I|=\varepsilon_I}\frac{dz}{2\pi i}\partial\left(\ln\varphi_{\rm cl}\right)\right]$$

- N delta functions for a_r $(r=1,\cdots,N)$.
- $\sum a_r=0,\cdots$ guarantees that $\varphi_{\rm cl}(z)\,dz^2$ is regular at $z=\infty$. $\oint_{C_{\mathcal{I}}}\frac{dz}{2\pi i}\sqrt{\varphi_{\rm cl}}-1=0$ $(\mathcal{I}=1,\cdots N-3)$ are satisfied by ϕ. go
- However, we are not sure if these delta functions alone fix a_r uniquely.

φ_{cl} coincides with ϕ

- Since Rem

$$\lim_{\varepsilon_I\to 0}\oint_{|z-z_I|=\varepsilon_I}\frac{dz}{2\pi i}\partial\left(\ln\varphi_{\mathrm{cl}}\right)=\begin{cases}1 & \text{if } \varphi_{\mathrm{cl}}\left(z_I\right)=0\\ 0 & \text{otherwise}\end{cases},$$

$\varphi_{\mathrm{cl}}\left(z\right)$ is fixed uniquely as Rem

$$\varphi_{\mathrm{cl}}\left(z\right)=\frac{A\prod_I\left(\left(z-z_I\right)\left(z-\bar{z}_I\right)\right)}{\prod_r\left(z-Z_r\right)^2},$$

$$A=\frac{\prod_{s\neq r}\left(Z_r-Z_s\right)^2}{\prod_I\left(\left(Z_r-z_I\right)\left(Z_r-\bar{z}_I\right)\right)}\ \text{for all } r,$$

which coincides with the quadratic differential ϕ. back

Remark

- Since $\oint_{|z-z_I|=\varepsilon_I} \frac{dz}{2\pi i}\partial\left(\ln\varphi_{\text{cl}}\right)$ is not a delta function, these alone cannot fix a_r.
- $\delta\left(\varphi\left(z_I\right)\right)$ does not commute with $T\left(z\right)$. back

Remark

$$T(z)\,\partial(\ln\varphi)(z') \sim \frac{4}{(z-z')^3} + \frac{1}{(z-z')^2}\partial(\ln\varphi)(z') + \frac{1}{z-z'}\partial^2(\ln\varphi)(z')$$

$$\left[T(z)\,,\, \oint_{z_I} \frac{dz'}{2\pi i}\partial(\ln\varphi)(z')\right] = 0$$

back

Remark

If one chooses sufficiently small ε_I $(I=1,\cdots,N-2)$

$$
\begin{aligned}
&\int [d\delta\varphi d\tilde{\varpi}] \exp\left(-\frac{1}{\pi}\int d^2z\delta\varphi\bar{\partial}\tilde{\varpi}\right) \\
&\quad\times\int\prod_r da_r\delta\left(\sum_r a_r\right)\delta\left(\sum_r (a_rZ_r+1)\right)\delta\left(\sum_r (a_rZ_r^2+2Z_r)\right) \\
&\qquad\times\prod_{\mathcal{I}}\delta\left(\oint_{C_{\mathcal{I}}}\frac{dz}{2\pi i}\sqrt{\varphi_{\mathrm{cl}}}-1\right)\prod_I\lim_{\varepsilon_I\to 0}\oint_{|z-z_I|=\varepsilon_I}\frac{dz}{2\pi i}\partial\left(\ln\varphi_{\mathrm{cl}}\right) \\
&=\int [d\delta\varphi d\tilde{\varpi}] \exp\left(-\frac{1}{\pi}\int d^2z\delta\varphi\bar{\partial}\tilde{\varpi}\right) \\
&\quad\times\int\prod_r da_r\delta\left(\sum_r a_r\right)\delta\left(\sum_r (a_rZ_r+1)\right)\delta\left(\sum_r (a_rZ_r^2+2Z_r)\right) \\
&\qquad\times\prod_{\mathcal{I}}\delta\left(\oint_{C_{\mathcal{I}}}\frac{dz}{2\pi i}\sqrt{\varphi_{\mathrm{cl}}}-1\right)\prod_I\oint_{|z-z_I|=\varepsilon_I}\frac{dz}{2\pi i}\partial\left(\ln\varphi_{\mathrm{cl}}\right)
\end{aligned}
$$

back

Remark

From the action

$$\frac{1}{\pi}\int d^2z\left(\varphi\bar{\partial}\varpi + \text{c.c.}\right) + \frac{d-26}{24}S_{\text{Liouville}}\left[\ln\sqrt{\varphi\bar{\varphi}}\right]$$

we can show

$$\varphi(z)\,\varphi(w) \sim \text{regular}\,,$$

back

Remark

- Since

$$\partial\varphi_{\mathrm{cl}}\left(z_I\right) = \frac{A\prod_{J\neq I}\left(z_I - z_J\right)\prod_J\left(z_I - \bar{z}_J\right)}{\prod_r\left(z_I - Z_r\right)^2},$$

$$A = \frac{\prod_{s\neq r}\left(Z_r - Z_s\right)^2}{\prod_I\left(\left(Z_r - z_I\right)\left(Z_r - \bar{z}_I\right)\right)} \text{ for all } r$$

$(\partial\varphi)^{-\frac{1}{3}\left(\frac{5}{8}\alpha^2+\frac{3}{4}\alpha\right)}$ is well-defined around $z = z_I$ for generic configuration

back