Multiloop amplitudes of light-cone gauge bosonic string field theory in noncritical dimensions

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Based on

Nobuyuki Ishibashi and K.M.

work in preparation (arXiv:1110.????)

"SFT 2011" at Prague, September 20, 2011

Although this work concentrates on LC gauge bosonic SFT,

the motivation resides in (NSR) super SFT.

This work is a part of a series of our works on dimensional regularization of the LC gauge super SFT:

Baba-Ishibashi-K.M.
JHEP **10** (2009) 035, ibid. **12** (2009) 010,
ibid. **08** (2010) 102, ibid. **01** (2010) 119.

Ishibashi-K.M.
JHEP **01** (2011) 008, ibid. **07** (2011) 090.

Why Light-cone gauge SFT again now?

Closed (type II) superstring field theory has not been studied so much

in contrast to several attempts in the open string case.

- WZW type theory (Berkovits),
- modified cubic theory (Arefeva-Medvedev-Zubarev, Preitschopf-Thorn-Yost), etc

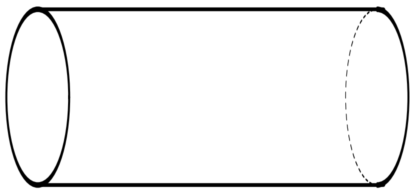
The light-cone gauge formulation (Kaku-Kikkawa, Sin)

$\Longrightarrow$ of simple form and thus convenient to define the theory

It should be important to reconsider the LC gauge closed super SFT.

Light-cone gauge string field theory

$$S = \int dt \left[\frac{1}{2} \Phi \cdot \left(2p^+ i \frac{\partial}{\partial t} - \left(L_0^{\rm LC} + \tilde{L}_0^{\rm LC} - \frac{c}{12} \right) \right) \Phi + \frac{2g}{3} \Phi^3 \right]$$



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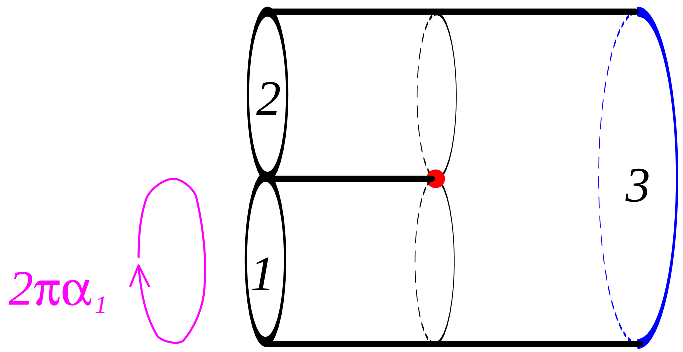

- $\alpha_r = 2p_r^+$: string-length parameter, $c = d - 2$
- $\Phi \left[t, p^+, X^i(\sigma) \right]$: string field

Light-cone gauge string field theory

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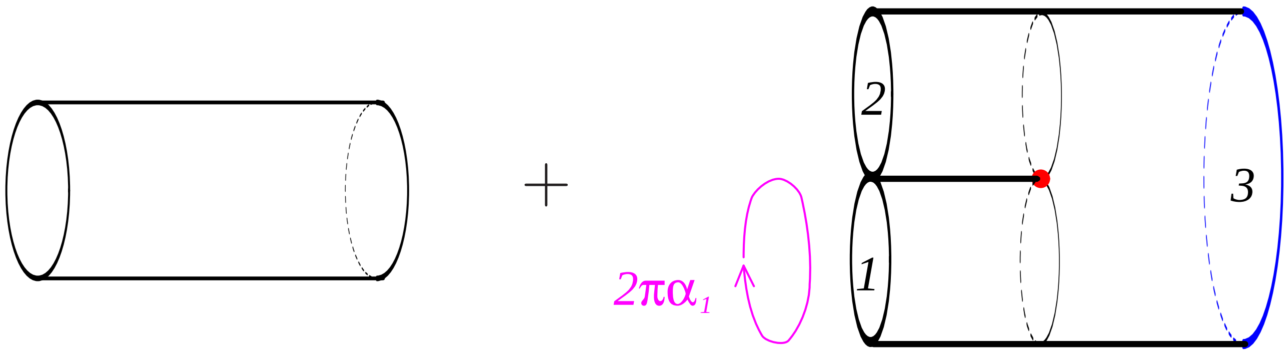


- $\alpha_r = 2p_r^+$: string-length parameter, $c = \frac{3}{2}\hat{c} = \frac{3}{2}(d-2)$
- $\Phi\left[t, p^+, X^i(\sigma); \psi^i(\sigma), \tilde{\psi}^i(\sigma)\right]$: string field

superstring case

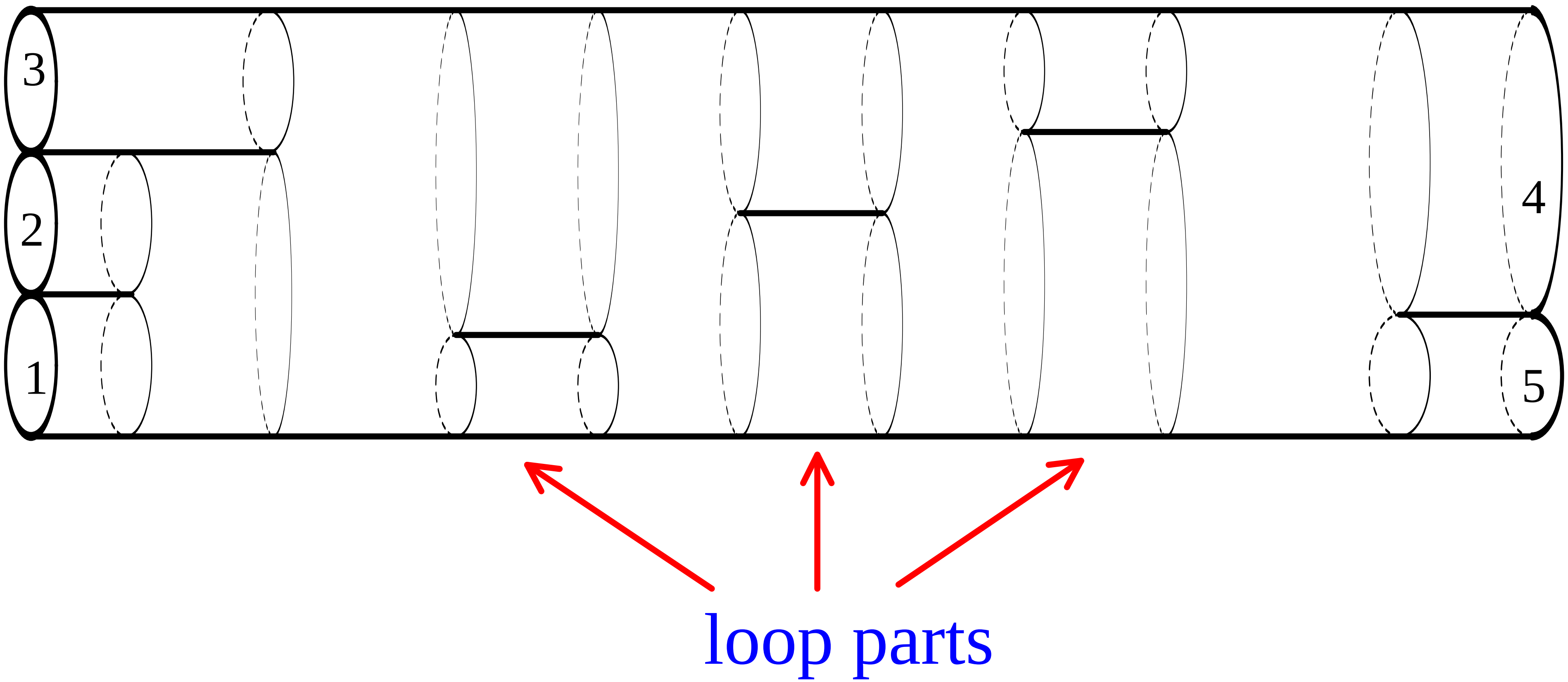
- $T_F^{\mathrm{LC}}\tilde{T}_F^{\mathrm{LC}}$ must be inserted at interaction point

$\Leftarrow$ Lorentz invariance for $d = 10$

(Mandelstam ('74), S.-J. Sin ('89) ...)

▶ light-cone string diagram

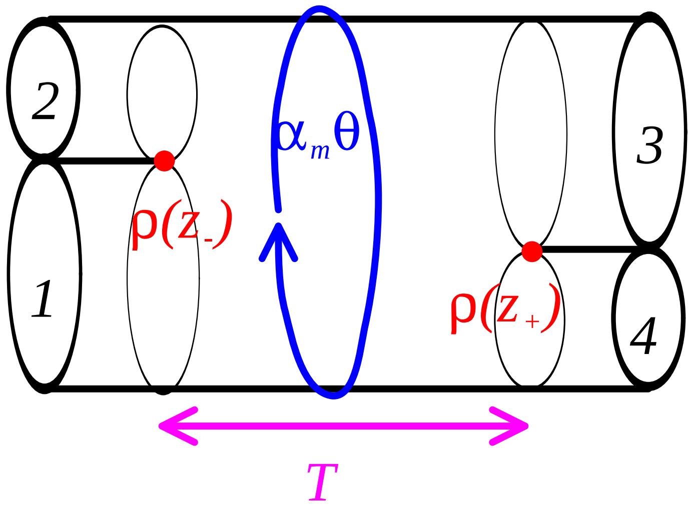
obtained by connecting interaction vertices using propagators



Motivation: Divergences caused by colliding T_F^{LC} in superstring case

e.g. 4pt amplitudes

$$\mathcal{A}_4 = \int d\mathcal{T} d\bar{\mathcal{T}}$$



- $\mathcal{T} = T + i\alpha_m\theta$
- At $\mathcal{T} = 0$, unwanted divergence

$$T_F^{\mathrm{LC}}(z_+)T_F^{\mathrm{LC}}(z_-) \sim \frac{\frac{3}{2}(d-2)}{(z_+ - z_-)^3}$$

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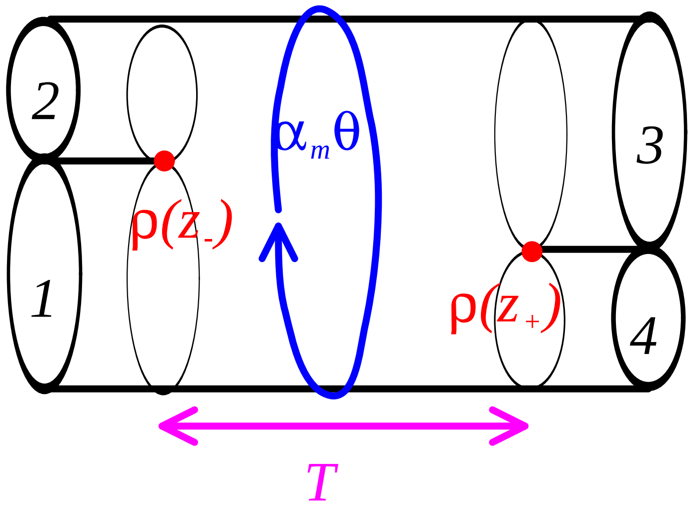

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Some regularization is necessary even at tree level

What we have done so far

- proposed to apply dimensional regularization to SFT (Baba-Ishibashi-K.M.)
  - Our scheme

(1) Formulate LC gauge SFT in $d \neq 10$

(2) take d (central charge $\hat{c} = d - 2$) to be a large negative value

(3) analytic continuation $d \to 10$ ($\hat{c} \to 8$) in the end

So far, we have verified our scheme for the amplitudes at tree-level (even involving the strings in Ramond sector).

(Baba-Ishibashi-K.M.)

- shown that our scheme indeed regularizes the divergences of the amplitudes at the tree-level (even involved with the Ramond sector)

key ingredient

Anomaly contribution, i.e. the Liouville action $\Gamma[\phi]$, from the conformal factor ϕ of the worldsheet metric

$$ds^2 = e^{\phi} dz d\bar{z}$$

serves as a regularization factor

with $\hat{c} \equiv d-2$ taken to be sufficiently large negative.

In fact,

$$e^{-\frac{\hat{c}}{16}\Gamma[\phi]} \sim |z_I - z_J|^{-\frac{\hat{c}}{8}} \qquad (z_I \sim z_J)$$

z_I, z_J: interaction points of the light-cone string diagram

- NOTE

Light-cone gauge formulation in noncritical dimensions

$\Longrightarrow$ We give up the Lorentz invariance in the regularization.

We are satisfied if the Lorentz symmetry is recovered in the limit $d \to 10 \quad (\hat{c} \to 8)$ taken in the end of the computations.

Question

Is dimensional regularization compatible with gauge symmetry of SFT?

Difficult at the second quantized level.

gauge invariant SFT for $d \neq 10$ or $d \neq 26$	(??)

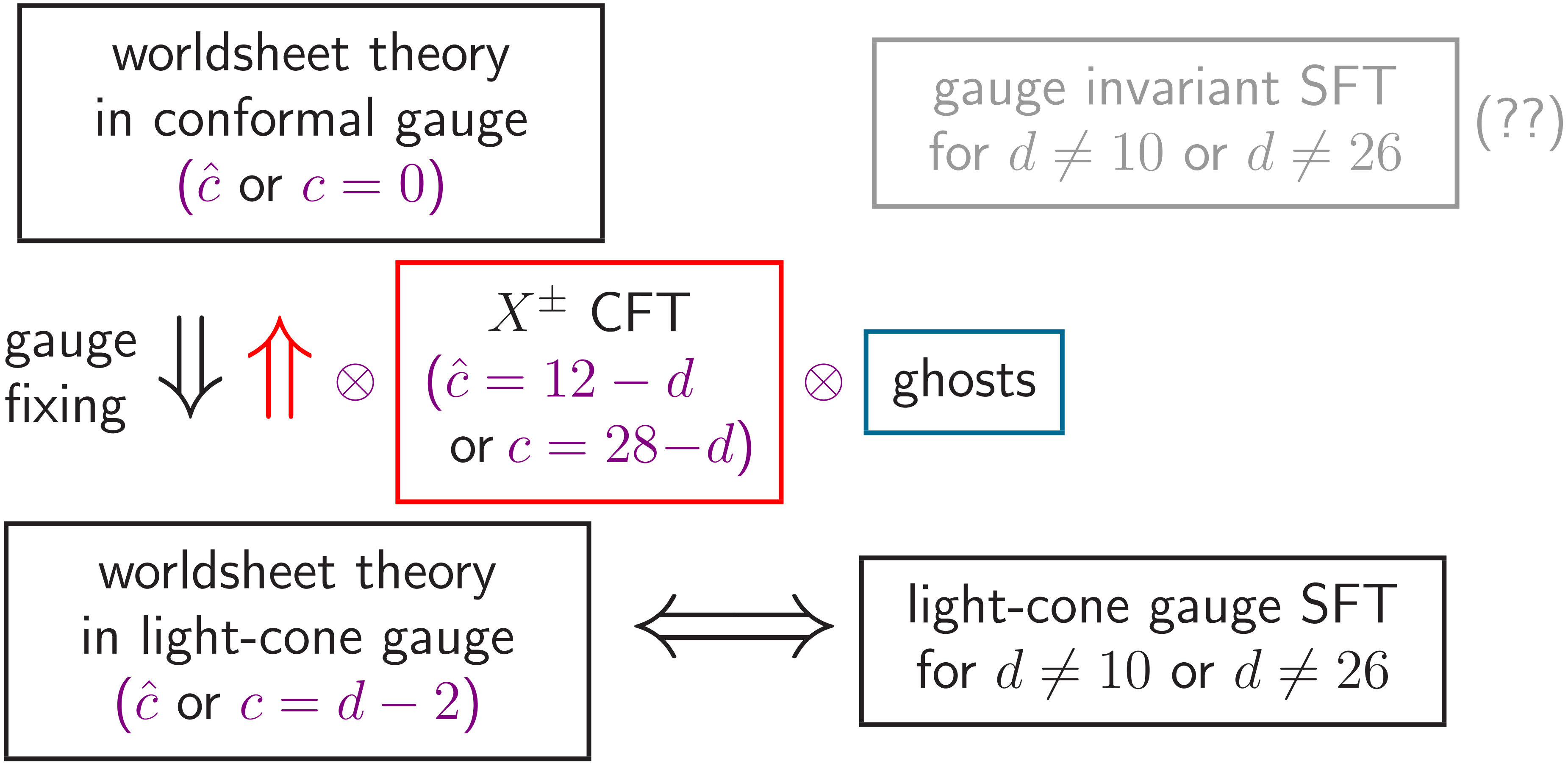
$$\Downarrow \text{ gauge fixing (??)}$$

light-cone gauge SFT for $d \neq 10$ or $d \neq 26$

This has not been achieved even for $d = 26$.

- We have constructed the CFT for the longitudinal variables ("$X^{\pm}$ CFT") s.t. at the first quantized level

  the tree amplitudes of the LC gauge SFT $\rightarrow$ BRST invariant form



(Proved in critical dimensions (D'Hoker-Giddings, Aoki-D'Hoker-Phong))

The question that we would like to consider:

Our dimensional regularization scheme and the $X^{\pm}$ CFT do the job at loop-levels as well?

As a first step towards this problem in the NSR superstring theory, in this work

we restrict ourselves to bosonic strings

What we did in this work

In the multiloop amplitudes
of LC gauge bosonic SFT for $d \neq 26$,

- evaluate the anomaly contribution $e^{-\frac{d-2}{24}\Gamma}$
- show the modular invariance
- describe $e^{-\frac{d-2}{24}\Gamma}$ in terms of the correlation function of the $X^{\pm}$ CFT and the bc-ghost system,

  $\Rightarrow$ rewrite the amplitudes into a BRST invariant form

$\Longleftarrow$ Techniques developed for tree-level amplitudes are directly applicable.

Plan of the talk

§1. Introduction

§2. Anomaly contribution $e^{-\frac{d-2}{24}\Gamma}$ and modular invariance

§3. $X^{\pm}$ CFT, bc ghosts and BRST invariant form of amplitudes

§4. Summary and discussions

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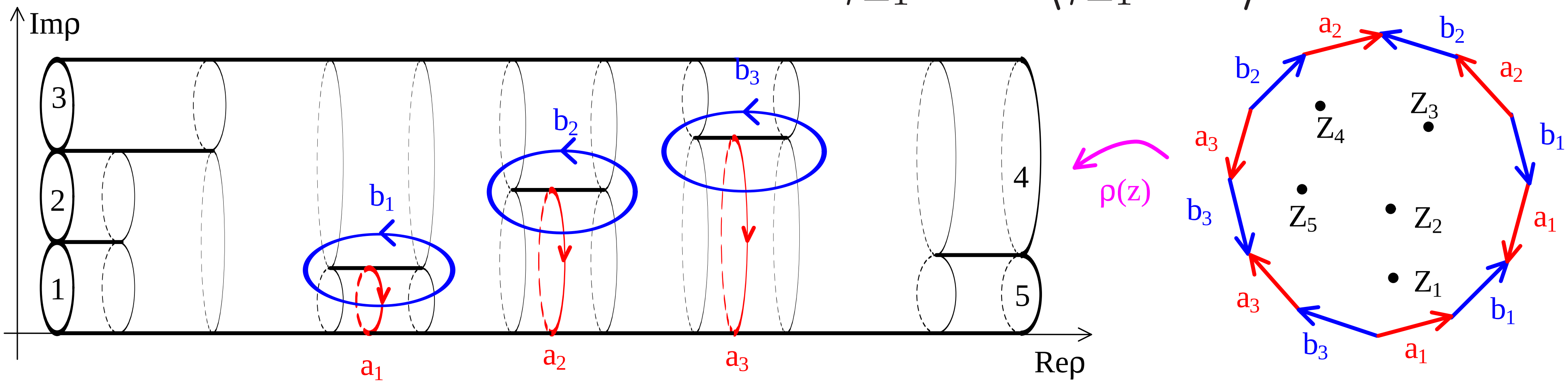
h-loop N-string amplitudes

$$\mathcal{A}_N = \int [dT][\alpha d\alpha][d\theta](2\pi)^2\delta^2\left(\sum_{r=1}^{N} p_r^{\pm}\right)\left\langle \prod_{r=1}^{N} V_r^{\mathrm{LC}} \right\rangle e^{-\frac{d-2}{24}\Gamma}$$



- Mandelstam mapping

$$\rho(z) = \sum_{r=1}^{N} \alpha_r \left[\ln E(z, Z_r) - 2\pi i \int_{P_0}^{z} \omega \frac{1}{\mathrm{Im}\,\Omega}\, \mathrm{Im} \int_{P_0}^{Z_r} \omega \right]$$

$\Bigg($ *cf.* For tree diagram $\quad \rho(z) = \sum_{r=1}^{N} \alpha_r \ln(z - Z_r) \Bigg)$

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- $\omega = (\omega_1, \ldots, \omega_h)$: a basis of the holomorphic one-forms on the genus h Riemann surface

$$\oint_{a_j} \omega_k = \delta_{j,k} \;, \qquad \oint_{b_j} \omega_k = \Omega_{jk} \leftarrow \text{period matrix}$$

- prime form

$$E(z, w) \equiv \frac{\theta[s]\left(\int_w^z \omega \middle| \Omega\right)}{\sqrt{\sum_{j=1}^{h} \frac{\partial\theta[s]}{\partial\zeta_j}(0|\Omega)\omega_j(z)}\sqrt{\sum_{j=1}^{h} \frac{\partial\theta[s]}{\partial\zeta_j}(0|\Omega)\omega_j(w)}}$$

$\theta[s](\zeta|\Omega)$: theta function for $\zeta \in \mathbb{C}^h/(\mathbb{Z}^h\Omega + \mathbb{Z}^h)$

with an odd spin structure $[s]$

$$\rho(z) = \sum_{r=1}^{N} \alpha_r \left[\ln E(z, Z_r) - 2\pi i \int_{P_0}^{z} \omega \frac{1}{\mathrm{Im}\,\Omega} \mathrm{Im} \int_{P_0}^{Z_r} \omega \right]$$

- $\omega = (\omega_1, \ldots, \omega_h)$: a basis of the holomorphic one-forms on the genus h Riemann surface

$$\oint_{a_j} \omega_k = \delta_{j,k} \ , \qquad \oint_{b_j} \omega_k = \Omega_{jk} \leftarrow \text{period matrix}$$

- prime form

$$E(z, w) \sim z - w \qquad \text{for } z \sim w$$

$$\left(\textit{cf.} \text{ For tree diagram} \quad \rho(z) = \sum_{r=1}^{N} \alpha_r \ln(z - Z_r) \right)$$

Mandelstam mapping $\rho(z)$ is determined by demanding that

- □ $d\rho = dz\partial\rho(z)$ is a 1-form globally well-defined, namely single-valued, on the LC string diagram.

- □ $\partial\rho(z) \sim \dfrac{\alpha_r}{z - Z_r}$ for $z \sim Z_r$, and regular everywhere else.

- □ The light-cone time $\mathrm{Re}\,\rho(z)$ is single valued on the string diagram, while $\mathrm{Im}\,\rho(z)$ is not. $\Longrightarrow$ $\mathrm{Re}\oint_{a_j} d\rho = \mathrm{Re}\oint_{b_j} d\rho = 0$

$\Longrightarrow$ $\partial\rho(z)$ has $2h - 2 + N$ zeros z_I $(I = 1, \ldots, 2h - 2 + N)$

They correspond to the $2h - 2 + N$ interaction points $\rho(z_I)$ of the LC string diagram.

Evaluation of $\exp\left[-\frac{d-2}{24}\Gamma\right]$

- worldsheet metric

$$ds^2 = d\rho d\bar{\rho} = e^{\phi} dz d\bar{z} \ , \qquad \phi = \ln |\partial\rho(z)|^2$$

- Liouville action

$$\Gamma[\phi] = -\frac{24}{24\pi}\int d^2z\, \partial\phi\bar{\partial}\phi$$

$$\phi \text{ is singular at } \begin{cases} \text{the zeros } z_I \text{ (interaction points)} \\ \text{the poles } Z_r \text{ (punctures for external legs)} \end{cases} \text{ of } \partial\rho(z)$$

$$\Downarrow$$

Evaluation of Γ is not straightforward but needs some care.

(Mandelstam)

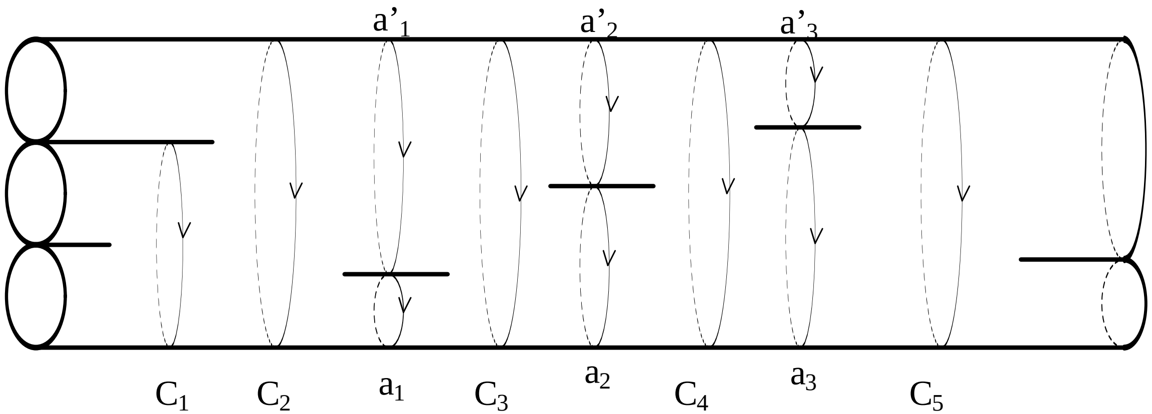
- Evaluate the variation of $\ln$(partition function) w.r.t. moduli parameters:

$$\delta \ln \left[\left(\frac{8\pi^2 \det{}'(-\nabla^2)}{\int d^2 z \sqrt{g}} \right)^{-\frac{d-2}{2}} e^{-\frac{d-2}{24}\Gamma} \right]$$

$$= \sum_{\mathcal{I}} \delta \mathcal{T}_{\mathcal{I}} \oint_{C_{\mathcal{I}}} \frac{d\rho}{2\pi i} \left\langle T^{\mathrm{LC}}(\rho) \right\rangle + \sum_{j=1}^{h} \delta \mathcal{T}_j \oint_{a_j + a'_j} \frac{d\rho}{2\pi i} \left\langle T^{\mathrm{LC}}(\rho) \right\rangle + \mathrm{c.c.}$$

$\mathcal{T}$'s: complex moduli parameters for internal propagators

$$\mathcal{T} = T + i\alpha\theta$$



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$\mathcal{T}$'s: complex moduli parameters for internal propagators

$$\mathcal{T} = T + i\alpha\theta$$

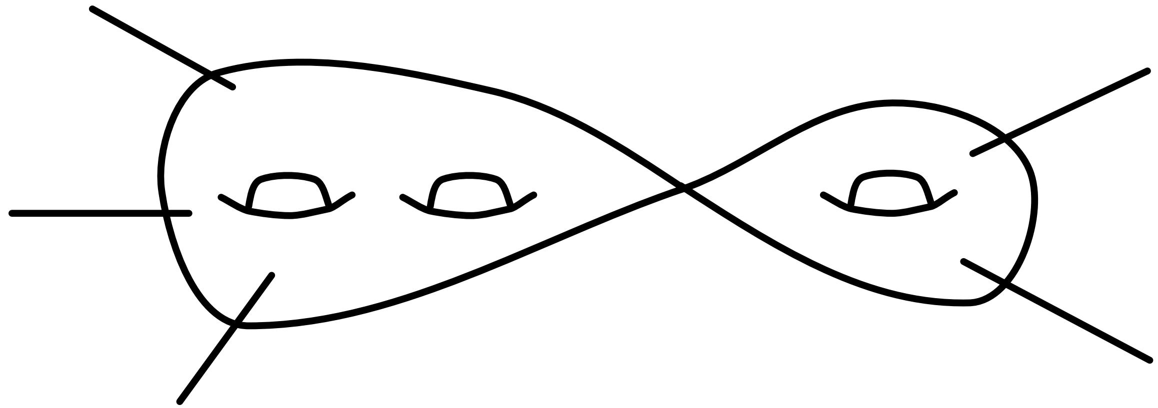
• $T^{\mathrm{LC}}(\rho) = (\partial\rho(z))^{-2} \left(T^{\mathrm{LC}}(z) - \frac{d-2}{12}\{\rho, z\} \right)$

The part of the Schwarzian derivative $\{\rho, z\}$ contributes to the anomaly factor $e^{-\frac{d-2}{24}\Gamma}$

- Notes:

  We do not take the variation w.r.t. the moduli associated with the b-cycles for simplicity.

  $\Leftarrow$ The part undetermined by such limited variations can be fixed by requiring that $e^{-\frac{d-2}{24}\Gamma}$ should correctly factorize in the degenerate limit of the Riemann surfaces.



Result

$$e^{-\Gamma} = e^{-\mathcal{W}} \, e^{-2\sum_r \mathrm{Re}\, \bar{N}^{rr}_{00}} \prod_I \left|\partial^2 \rho(z_I)\right|^{-3}$$

$$e^{-\mathcal{W}} \equiv \frac{\prod_I |\sigma(z_I)|^8}{\prod_r |\sigma(Z_r)|^8} \frac{\prod_{I<J} |E(z_I, z_J)|^4 \prod_{r<s} |E(Z_r, Z_s)|^4}{\prod_{I,r} |E(z_I, Z_r)|^4} \; e^{4\pi v \frac{1}{\mathrm{Im}\,\Omega} v}$$

$$v \equiv \mathrm{Im}\left(\sum_I \int_{P_0}^{z_I} \omega - \sum_r \int_{P_0}^{Z_r} \omega - 2\Delta \right) \; \in \; \mathbb{C}^h$$

$$\sigma(z) \equiv \exp\left[-\sum_{j=1}^{h} \oint_{a_j} \omega_j(w) \ln E(w, z) \right]$$

$\Delta \in \mathbb{C}^h$: vector of Riemann constants $\quad \Delta_j \equiv -\frac{\Omega_{jj}}{2} + \frac{1}{2} + \sum_{k \neq j} \oint_{a_k} \omega_k(z) \int_{P_0}^{z} \omega_j$

$\bar{N}^{rr}_{00}$: Neumann coefficient

Modular invariance

Under the modular transformations

$$\omega \mapsto \tilde{\omega} = \omega \frac{1}{C\Omega + D} \ , \quad \Omega \mapsto \tilde{\Omega} = (A\Omega + B) \frac{1}{C\Omega + D} \ ,$$

A, B, C, D: $h \times h$ integral matrices s.t. $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2h, \mathbb{Z})$,

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- $\rho(z) \;\mapsto\; \tilde{\rho}(z) = \rho(z) + i\pi \sum_{r=1}^{N} \alpha_r \int_{P_0}^{Z_r} \omega \frac{1}{C\Omega + D} C \int_{P_0}^{Z_r} \omega$
- $\sum_{r=1}^{N} \mathrm{Re}\, \bar{N}_{00}^{rr}$: modular invariant
- $e^{-\frac{d-2}{24}\Gamma}$: modular invariant

Modular invariance

Under the modular transformations

$$\omega \mapsto \tilde{\omega} = \omega \frac{1}{C\Omega + D} \,, \quad \Omega \mapsto \tilde{\Omega} = (A\Omega + B)\frac{1}{C\Omega + D} \,,$$

A, B, C, D: $h \times h$ integral matrices s.t. $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2h, \mathbb{Z})$,

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- $\sum_{r=1}^{N} \mathrm{Re}\, \bar{N}_{00}^{rr}$: modular invariant
- $e^{-\frac{d-2}{24}\Gamma}$: modular invariant

$\Longrightarrow$ **Amplitudes are modular invariant**

§1. Introduction

§2. Anomaly contribution $e^{-\frac{d-2}{24}\Gamma}$ and modular invariance

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§4. Summary and discussions

Action of $X^{\pm}$ CFT

$$S_{\pm} = -\frac{1}{2\pi}\int d^2z\,\left(\partial X^+\bar\partial X^- + \bar\partial X^+\partial X^-\right) + \frac{d-26}{24}\Gamma[\hat\phi]$$

$$\Gamma[\hat\phi] = -\frac{1}{\pi}\int d^2z\,\partial\hat\phi\bar\partial\hat\phi\,, \qquad \hat\phi \equiv \ln\left(-4\partial X^+\bar\partial X^+\right)$$

- energy-momentum tensor

$$\boxed{T_{X^{\pm}} = \partial X^+\partial X^- - \frac{d-26}{12}\{X^+\,,\,z\}}$$

$$\{X^+\,,\,z\} \equiv \frac{\partial^3 X^+}{\partial X^+} - \frac{3}{2}\left(\frac{\partial^2 X^+}{\partial X^+}\right)^2 \qquad \text{(Schwarzian)}$$

- OPE $\quad X^+X^+ \sim \text{regular}\,, \qquad \partial X^+(z)\partial X^-(z') \sim \frac{1}{(z-z')^2}\,,$

$$\partial X^-(z)\partial X^-(z') \sim -\frac{d-26}{12}\partial_z\partial_{z'}\left[\frac{1}{(z-z')^2}\frac{1}{\partial X^+(z)\partial X^+(z')}\right]$$

Expectation value of X^+

$S_\pm$ contains $\frac{1}{\partial X^+}$

$\Rightarrow$ ∂X^+ should have a non-zero expectation value

$\Rightarrow$ always with insertion $\prod_{r=1}^N e^{-ip_r^+ X^-}(Z_r, \bar{Z}_r)$ $\quad (\sum_{r=1}^N p_r^+ = 0)$

Consider the correlation function of the form

$$\left\langle F[X^+, X^-] \prod_{r=1}^N e^{-ip_r^+ X^-}(Z_r, \bar{Z}_r) \right\rangle_\pm$$

$$\equiv \int [dX^\pm] e^{-S_\pm} F[X^+, X^-] \prod_{r=1}^N e^{-ip_r^+ X^-}(Z_r, \bar{Z}_r)$$

X^+ indeed has an expectation value ($\rightarrow$ next slide)

For $F[X^+]$ (independent of X^-)

$$\left\langle F[X^+] \prod_{r=1}^{N} e^{-ip_r^+ X^-}(Z_r, \bar{Z}_r) \right\rangle_{\pm}$$

For $F[X^+]$ (independent of X^-)

Using $2\pi \sum_{r=1}^{N} p_r^+ \delta^2(z - Z_r) = \partial\bar{\partial}\left(\rho(z) + \bar{\rho}(\bar{z})\right)$,

$$\left\langle F[X^+] \prod_{r=1}^{N} e^{-ip_r^+ X^-}(Z_r, \bar{Z}_r) \right\rangle_{\pm}$$

$$= \int [dX^{\pm}] e^{-\frac{1}{\pi}\int d^2z\, X^- \partial\bar{\partial}\left(X^+ + \frac{i}{2}(\rho(z)+\bar{\rho}(\bar{z}))\right)} F[X^+]\, e^{-\frac{d-26}{24}\Gamma[\ln(-4\partial X^+ \bar{\partial} X^+)]}$$

For $F[X^+]$ (independent of X^-)

Using $2\pi \sum_{r=1}^{N} p_r^+ \delta^2(z - Z_r) = \partial\bar{\partial}\,(\rho(z) + \bar{\rho}(\bar{z}))$,

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$$\sim \int [dX^+]\, \delta\left(X^+ + \frac{i}{2}(\rho + \bar{\rho})\right) F[X^+]\, e^{-\frac{d-26}{24}\Gamma[\ln(-4\partial X^+ \bar{\partial} X^+)]}$$

$$= F\left[-\frac{i}{2}(\rho + \bar{\rho})\right] e^{-\frac{d-26}{24}\Gamma[\ln(\partial\rho\bar{\partial}\bar{\rho})]}$$

$\Rightarrow$ X^+ has an expectation value

$$X^+(z, \bar{z}) \sim -\frac{i}{2}(\rho(z)+\bar{\rho}(\bar{z})) \quad (\leftarrow \text{Lorentzian light-cone time})$$

$\sim$ light-cone gauge

$F[-\frac{i}{2}(\rho+\bar\rho)]e^{-\frac{d-26}{24}\Gamma[\ln(\partial\rho\bar\partial\bar\rho)]}$ as a generating functional of correlation functions involving X^-

$$
\begin{aligned}
&\left\langle F[X^+] X^-(Z_N, \bar Z_N) \prod_{r=1}^{N-1} e^{-ip_r^+ X^-}(Z_r, \bar Z_r) \right\rangle_\pm \\
&\sim i\partial_{p_N^+} \left\langle F[X^+] \prod_{r=1}^{N} e^{-ip_r^+ X^-}(Z_r, \bar Z_r) \right\rangle_\pm \Bigg|_{p_N^+=0} \\
&= i\partial_{p_N^+} \left(F\left[-\frac{i}{2}(\rho+\bar\rho)\right] e^{-\frac{d-26}{24}\Gamma[\ln(\partial\rho\bar\partial\bar\rho)]} \right) \Bigg|_{p_N^+=0}
\end{aligned}
$$

- We can go on $\rightarrow$ correlation functions with more than one X^-

$\Longleftrightarrow$ The method developed for the tree level amplitudes are directly applicable.

expression of V_r^{LC} in terms of the DDF vertex operator V_r^{DDF}

$$(2\pi)^2\delta^2\left(\sum_{r=1}^{N} p_r^{\pm}\right)\prod_{r=1}^{N} V_r^{\mathrm{LC}}$$

$$= \prod_{r=1}^{N}\left(\alpha_r e^{2\,\mathrm{Re}\,\bar{N}_{00}^{rr}}\right) e^{\frac{d-26}{24}\Gamma}\left\langle \prod_{r=1}^{N} V_r'^{\mathrm{DDF}}(Z_r,\bar{Z}_r) e^{\frac{d-26}{24}\frac{i}{p_r^+}X^+}\left(z_{I(r)},\bar{z}_{I(r)}\right)\right\rangle_{X^\pm}$$

- The factors in purple are peculiar to the case in noncritical dimensions.
- $V_r^{\mathrm{LC}} \quad \leftrightarrow \quad V_r^{\mathrm{DDF}}$

$\Big($ • $V_r'^{\mathrm{DDF}} =: V_r^{\mathrm{DDF}} e^{-\frac{d-26}{24}\frac{i}{p_r^+}X^+}: \quad \longleftarrow$ weight $(1,1)$ $\Big)$

expression of $e^{-\Gamma}$ in terms of correlation function of bc-ghost CFT

ghost correlation function (Verlinde-Verlinde, Alvarez-Gaume-Bost-Moore-Vafa, ...)

$$\left\langle \prod_{i=1}^{M+3(h-1)} b(z_i) \prod_{j=1}^{M} c(w_j) \right\rangle = \left(\frac{8\pi^2 \det'\left(-\nabla^2\right)}{\int d^2 z \sqrt{g} \det \mathrm{Im}\,\Omega} \right)^{-\frac{1}{4}} \theta \left(\sum_i \int_{P_0}^{z_i} \omega - \sum_j \int_{p_0}^{w_j} \omega - 3\Delta \middle| \Omega \right)$$

$$\times \frac{\prod_{i<\acute{\imath}} E(z_i, z_{\acute{\imath}}) \prod_{j<\acute{\jmath}} E(w_j, w_{\acute{\jmath}})}{\prod_{i,j} E(z_i, w_j)} \frac{\prod_i \sigma(z_i)^3}{\prod_j \sigma(w_j)^3}$$

expression of $e^{-\Gamma}$ in terms of correlation function of bc-ghost CFT

ghost correlation function (Verlinde-Verlinde, Alvarez-Gaume-Bost-Moore-Vafa, ...)

$$\left\langle \prod_{i=1}^{M+3(h-1)} b(z_i) \prod_{j=1}^{M} c(w_j) \right\rangle = \left(\frac{8\pi^2 \det'(-\nabla^2)}{\int d^2 z \sqrt{g} \det \mathrm{Im}\,\Omega} \right)^{-\frac{1}{4}} \theta\left(\sum_i \int_{P_0}^{z_i} \omega - \sum_j \int_{p_0}^{w_j} \omega - 3\Delta \middle| \Omega \right)$$
$$\times \frac{\prod_{i<\acute{\imath}} E(z_i, z_{\acute{\imath}}) \prod_{j<\acute{\jmath}} E(w_j, w_{\acute{\jmath}})}{\prod_{i,j} E(z_i, w_j)} \frac{\prod_i \sigma(z_i)^3}{\prod_j \sigma(w_j)^3}$$

- Choose $M = N + 1$
  - $w_j = (Z_r, w)$, Z_r: N poles of $\partial\rho(z)$, $\forall w$
  - $z_i = (z_I, z_{\hat{\imath}})$, z_I: $2h - 2 + N$ zeros of $\partial\rho(z)$, $\forall z_{\hat{\imath}}$ $(\hat{\imath} = 1 \sim h)$

- express $\left| \left\langle \prod_{i=1}^{M+3(h-1)} b(z_i) \prod_{j=1}^{M} c(w_j) \right\rangle \right|^2$ and $|\partial\rho(z)|^2$ in terms of the Arakelov Green's function (Dugan-Sonoda, D'Hoker-Phong, ...)

expression of $e^{-\Gamma}$ in terms of correlation function of bc-ghost CFT

ghost correlation function (Verlinde-Verlinde, Alvarez-Gaume-Bost-Moore-Vafa, ...)

$$\left\langle \prod_{i=1}^{M+3(h-1)} b(z_i) \prod_{j=1}^{M} c(w_j) \right\rangle = \left(\frac{8\pi^2 \det'(-\nabla^2)}{\int d^2 z \sqrt{g} \det \mathrm{Im}\,\Omega} \right)^{-\frac{1}{4}} \theta\left(\sum_i \int_{P_0}^{z_i} \omega - \sum_j \int_{p_0}^{w_j} \omega - 3\Delta \middle| \Omega \right)$$
$$\times \frac{\prod_{i<\acute{\imath}} E(z_i, z_{\acute{\imath}}) \prod_{j<\acute{\jmath}} E(w_j, w_{\acute{\jmath}})}{\prod_{i,j} E(z_i, w_j)} \frac{\prod_i \sigma(z_i)^3}{\prod_j \sigma(w_j)^3}$$

For the zeros z_I's and the poles Z_r's of $\partial\rho(z)$; $\forall z_{\hat{\imath}}$, $\forall w$,

$$e^{-\Gamma} = \left(\frac{8\pi^2 \det'(-\nabla^2)}{\int d^2 z \sqrt{g}} \right)^{-1} \frac{\det \mathrm{Im}\,\Omega}{\left|\det_{\hat{\imath},\hat{\jmath}} \omega_{\hat{\imath}}(z_{\hat{\jmath}})\right|^2} \prod_{r=1}^{N} \left(|\alpha_r|^{-1} e^{-2\,\mathrm{Re}\,\bar{N}_{00}^{rr}} \right)$$
$$\times \left| \left\langle \prod_{I=1}^{2h-2+N} \frac{b}{\partial^2\rho}(z_I) \prod_{\hat{\imath}=1}^{h} \frac{b}{\partial\rho}(z_{\hat{\imath}}) \prod_{r=1}^{N} c(Z_r)\, (\partial\rho c)\,(w) \right\rangle \right|^2$$

BRST invariant form of the amplitudes

Putting the above re-expressions using the $X^{\pm}$ CFT and the bc ghosts together , we obtain

$$
\begin{aligned}
\mathcal{A}_N = & \int [dT] \int [\alpha d\alpha] \int [d\theta] \det \operatorname{Im} \Omega \\
& \times \left\langle \prod_{i=1}^{h} \left| \oint_{a_i} \frac{dz}{2\pi i} \frac{b}{\partial \rho}(z) \oint_{a'_i} \frac{dz}{2\pi i} \frac{b}{\partial \rho}(z) \right|^2 \prod_{\mathcal{I}=1}^{h-3+N} \left| \oint_{C_{\mathcal{I}}} \frac{dz}{2\pi i} \frac{b}{\partial \rho}(z) \right|^2 \right. \\
& \times \prod_{r=1}^{N} \oint_{z_{I(r)}} \frac{dz}{2\pi i} \partial \ln \partial X^+(z) \oint_{\bar{z}_{I(r)}} \frac{d\bar{z}}{2\pi i} \bar{\partial} \ln \bar{\partial} X^+(\bar{z}) e^{\frac{d-26}{24} \frac{i}{p_r^+} X^+}(z, \bar{z}) \\
& \left. \times \prod_{r=1}^{N} \left(c\tilde{c} V_r^{\prime \mathrm{DDF}} \right) (Z_r, \bar{Z}_r) \right\rangle
\end{aligned}
$$

$\longleftarrow$ BRST invariant

§1. Introduction

§2. Anomaly contribution $e^{-\frac{d-2}{24}\Gamma}$ and modular invariance

§3. $X^{\pm}$ CFT, bc ghosts and BRST invariant form of amplitudes

§4. Summary and discussions

Summary and discussions

☐ As a first step towards the dimensional regularization of the multiloop amplitudes in LC gauge NSR super SFT,

In the multiloop amplitudes of LC gauge bosonic SFT for $d \neq 26$,

- evaluate the anomaly contribution $e^{-\frac{d-2}{24}\Gamma}$
- show the modular invariance
- rewrite the scattering amplitudes into a BRST invariant form using the longitudinal variables in the $X^{\pm}$ CFT and the bc-ghosts

$\Longleftarrow$ Techniques developed for the tree-level amplitudes are directly applicable.

□ What we should do next is to supersymmetrize these analyses.

- multiloop supersheet
- treatment of the picture changing operator
- spacetime fermions, GSO projection

□ would like to construct gauge invariant super SFT

$\Uparrow$

based on the worldsheet CFT in the conformal gauge thus far formulated

$$\left[X^{\pm} \text{ CFT} \quad \oplus \quad \text{free } X^i, \psi^i, \tilde{\psi}^i \quad \oplus \quad \text{ghosts} \right]$$

$\Longrightarrow \alpha = p^+$ HIKKO type theory (??)