# Gauge-fixing conditions and propagators in open superstring field theory

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# Tentative Titles of the Papers

- "Open Superstring Field Theory I: Gauge fixing, ghost structure and propagator"

  by N. Berkovits, M. Kroyter, Y. Okawa, M. Schnabl, B. Zwiebach and S. T.

- "Validity of Gauge-Fixing Conditions and Structure of Propagators in Open Superstring Field Theory"

  by S. T.

## String Perturbation Theory

Closed strings appear in loop diagrams of open strings.

## (Covariant) String Field Theory (SFT)

It is non-trivial whether and how closed strings can be described in terms of open string FIELDS.

Can open SFT be consistently quantized without additional degrees of freedom such as closed string fields?

## Bosonic SFT

Divergences from $\begin{cases} \text{tadpole diagrams} \\ \text{tachyons} \end{cases}$

Quantization can be considered only in a formal way.

$\Rightarrow$ **Superstring Field Theory (SSFT)**

# Superstring Field Theory (SSFT)

- Witten's cubic SSFT
- Modified cubic SSFT
- Democratic SSFT

} with Picture-Changing Operators (PCOs)

- Wess-Zumino-Witten-like SSFT } without PCOs
- ...

We consider the **WZW-like SSFT**, concentrating on the NS sector.

Last year

• We need infinitely many ghosts and antighosts carrying various world-sheet ghost $\sharp$'s and picture $\sharp$'s.

• We "perturbatively" solved the master equation in the interacting theory, using the Batalin-Vilkovisky formalism.

## Today

- In my talk, we focus on the free WZW-like theory. (Michael and Nathan will talk about the interacting theory.)
- We propose two (or more, if time permits) gauge-fixing conditions.

One of them corresponds to the Siegel gauge in (unmodified) Witten's SSFT.

- We consider propagators.

# Plan

1. Gauge Fixing of the Free WZW-like SSFT
2. Relation to the Siegel Gauge in Witten's SSFT
3. Propagators
4. Another Gauge-Fixing Condition
5. Summary and Discussions

# 1. Gauge Fixing of the Free WZW-like SSFT

N. Berkovits, Nucl. Phys. B459 (1996) 439

$$S_0 = -\frac{1}{2}\int \Phi_{(0,0)}\big(Q\eta_0\Phi_{(0,0)}\big)\,, \quad \Phi_{(0,0)} \in \mathscr{H}_{\mathrm{large}}\,,$$

$$\delta\Phi_{(0,0)} = Q\Lambda_{(-1,0)} + \eta_0\Lambda_{(-1,1)} \quad (Q^2 = \eta_0^2 = \{Q,\eta_0\} = 0),$$

$Q$: BRST operator, Bosonization: $\beta \cong \mathrm{e}^{-\phi}\partial\xi\,,\ \gamma \cong \eta\mathrm{e}^{\phi}\,.$

$\Phi_{(g,p)}$: NS string field of world-sheet ghost number $g$ and picture $p$,

$$(g(Q),p(Q)) = (1,0)\,,\ (g(\eta_0),p(\eta_0)) = (1,-1)\,.$$

$\int(\cdots)$ vanishes unless the integrand $(\cdots)$ has $(g,p) = (2,-1)$.

$$S_0 = -\frac{1}{2}\int \Phi_{(0,0)}(Q\eta_0\Phi_{(0,0)}) = -\frac{1}{2}\langle\Phi_{(0,0)}|Q\eta_0|\Phi_{(0,0)}\rangle$$

$\delta\Phi_{(0,0)} = Q\Lambda_{(-1,0)} + \eta_0\Lambda_{(-1,1)}\,,$ GF condition: $\begin{bmatrix} b_0 \\ \xi_0 \end{bmatrix} |\Phi_{(0,0)}\rangle = 0$

FP action: $S_1 = -\Big(\langle B_{(3,-1)}|\, b_0 + \langle B_{(3,-2)}|\, \xi_0\Big)\Big(Q\,|\Phi_{(-1,0)}\rangle + \eta_0\,|\Phi_{(-1,1)}\rangle\Big),$

$B_{(3,*)}$: antighosts, $\Phi_{(-1,*)}$: ghosts.

Redefinition: $\langle\Phi_{(2,-1)}| := \langle B_{(3,-1)}|\, b_0 + \langle B_{(3,-2)}|\, \xi_0\,,$ $b_0\xi_0\,|\Phi_{(2,-1)}\rangle = 0\,.$

$$\Longrightarrow S_1 = -\langle\Phi_{(2,-1)}|\Big(Q\,|\Phi_{(-1,0)}\rangle + \eta_0\,|\Phi_{(-1,1)}\rangle\Big)$$

$$S_0 + S_1 = -\frac{1}{2}\langle\Phi_{(0,0)}|Q\eta_0|\Phi_{(0,0)}\rangle - \langle\Phi_{(2,-1)}|\left(Q\,|\Phi_{(-1,0)}\rangle + \eta_0\,|\Phi_{(-1,1)}\rangle\right)$$

gauge transf. of the ghosts

$$\delta\Phi_{(-1,0)} = Q\Lambda_{(-2,0)} + \eta_0\Lambda_{(-2,1)}$$
$$\delta\Phi_{(-1,1)} = Q\Lambda_{(-2,1)} + \eta_0\Lambda_{(-2,2)}$$

(Recall $Q^2 = \eta_0^2 = \{Q, \eta_0\} = 0$ .) GF condition $\begin{bmatrix} b_0 & 0 \\ \xi_0 & 0 \\ 0 & \xi_0 \end{bmatrix} \begin{bmatrix} |\Phi_{(-1,0)}\rangle \\ |\Phi_{(-1,1)}\rangle \end{bmatrix} = 0$

FP terms:

$$\begin{cases} -\langle B_{(4,-1)}|\,b_0\left(Q\,|\Phi_{(-2,0)}\rangle + \eta_0\,|\Phi_{(-2,1)}\rangle\right) \\ -\langle B_{(4,-2)}|\,\xi_0\left(Q\,|\Phi_{(-2,0)}\rangle + \eta_0\,|\Phi_{(-2,1)}\rangle\right) \\ -\langle B_{(4,-3)}|\,\xi_0\left(Q\,|\Phi_{(-2,1)}\rangle + \eta_0\,|\Phi_{(-2,2)}\rangle\right) \end{cases}$$

$$\begin{cases} B_{(4,*)}\text{: antighosts} \\ \Phi_{(-2,*)}\text{: ghosts} \end{cases}$$

Redefinition: $\begin{cases} \langle\Phi_{(3,-1)}| := \langle B_{(4,-1)}|\, b_0 + \langle B_{(4,-2)}|\, \xi_0 \,, \\ \langle\Phi_{(3,-2)}| := \langle B_{(4,-3)}|\, \xi_0 \,. \end{cases}$

$$\begin{bmatrix} b_0\xi_0 & 0 \\ 0 & \xi_0 \end{bmatrix} \begin{bmatrix} |\Phi_{(3,-1)}\rangle \\ |\Phi_{(3,-2)}\rangle \end{bmatrix} = 0 \,.$$

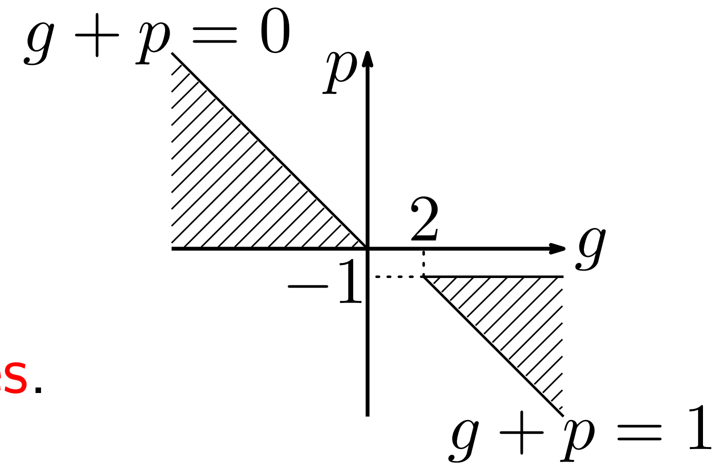
FP action:

$$S_2 = \langle\Phi_{(3,-1)}| \Big( Q\, |\Phi_{(-2,0)}\rangle + \eta_0\, |\Phi_{(-2,1)}\rangle \Big) + \langle\Phi_{(3,-2)}| \Big( Q\, |\Phi_{(-2,1)}\rangle + \eta_0\, |\Phi_{(-2,2)}\rangle \Big)$$

$$\text{ghosts:}\ \begin{cases} \Phi_{(-1,0)}\,,\ \Phi_{(-1,1)}\,, \\ \Phi_{(-2,0)}\,,\ \Phi_{(-2,1)}\,,\ \Phi_{(-2,2)}\,, \\ \Phi_{(-3,0)}\,,\ \Phi_{(-3,1)}\,,\ \Phi_{(-3,2)}\,,\ \Phi_{(-3,3)}\,, \\ \ldots \end{cases}$$

$$\text{antighosts:}\ \begin{cases} \Phi_{(2,-1)}\,, \\ \Phi_{(3,-1)}\,,\ \Phi_{(3,-2)}\,, \\ \Phi_{(4,-1)}\,,\ \Phi_{(4,-2)}\,,\ \Phi_{(4,-3)}\,, \\ \ldots \end{cases}$$

A string field $\Phi_{(g,\,p)}$ is admissible only when the lattice point $(g,\,p)$ belongs to the region with oblique lines.



# The gauge-fixed action:

$$\underbrace{-\frac{1}{2}\int\Phi_{(0,0)}(Q\eta_0\Phi_{(0,0)})}_{S_0}-\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\int\Phi_{(n+1,-m-1)}\left(Q\Phi_{(-n,m)}+\eta_0\Phi_{(-n,m+1)}\right)$$

$$\begin{bmatrix} b_0 \\ \xi_0 \end{bmatrix}\Phi_{(0,0)}=\begin{bmatrix} b_0 & 0 \\ \xi_0 & 0 \\ 0 & \xi_0 \end{bmatrix}\begin{bmatrix} \Phi_{(-1,0)} \\ \Phi_{(-1,1)} \end{bmatrix}=\begin{bmatrix} b_0 & 0 & 0 \\ \xi_0 & 0 & 0 \\ 0 & \xi_0 & 0 \\ 0 & 0 & \xi_0 \end{bmatrix}\begin{bmatrix} \Phi_{(-2,0)} \\ \Phi_{(-2,1)} \\ \Phi_{(-2,2)} \end{bmatrix}=\cdots=0\,,$$

$$b_0\xi_0\Phi_{(2,-1)}=\begin{bmatrix} b_0\xi_0 & 0 \\ 0 & \xi_0 \end{bmatrix}\begin{bmatrix} \Phi_{(3,-1)} \\ \Phi_{(3,-2)} \end{bmatrix}=\begin{bmatrix} b_0\xi_0 & 0 & 0 \\ 0 & \xi_0 & 0 \\ 0 & 0 & \xi_0 \end{bmatrix}\begin{bmatrix} \Phi_{(4,-1)} \\ \Phi_{(4,-2)} \\ \Phi_{(4,-3)} \end{bmatrix}=\cdots=0\,.$$

# 2. Relation to the Siegel Gauge in Witten's SSFT

Witten's free action: $S_0 = -\frac{1}{2}\int_{\mathrm{S}} \Psi_{(1,-1)}(Q\Psi_{(1,-1)})$

The gauge-fixed action: $S = -\frac{1}{2}\int_{\mathrm{S}} \Psi(Q\Psi)$ with constraints.

$$\Psi := \underbrace{\cdots + \Psi_{(-1,-1)} + \Psi_{(0,-1)}}_{\text{ghosts}} + \Psi_{(1,-1)} + \underbrace{\Psi_{(2,-1)} + \Psi_{(3,-1)} + \cdots}_{\text{antighosts}}.$$

In the Siegel gauge, where $b_0\Psi = 0$, we have

$$S = -\frac{1}{2}\int_{\mathrm{S}} \Psi(c_0 L_0)\Psi = \underbrace{-\frac{1}{2}\int_{\mathrm{S}} \Psi_{(1,-1)}(c_0 L_0)\Psi_{(1,-1)}}_{S_0} \underbrace{-\int_{\mathrm{S}} \Psi_{(2,-1)}(c_0 L_0)\Psi_{(0,-1)}}_{S_1}$$
$$+\cdots.$$

To see the relation between the two actions, we decompose $\Phi_{(g,p)}$ in the WZW-like theory, using $\{b_0,\, c_0\} = \{\eta_0,\, \xi_0\} = 1$.

$$\Phi_{(g,p)} = c_0\xi_0\Phi^{c\xi}_{(g,p)} + c_0\Phi^{c-}_{(g,p)} + \xi_0\Phi^{-\xi}_{(g,p)} + \Phi^{--}_{(g,p)}\,.$$

$\Phi^{c\xi}_{(g,p)}$, $\Phi^{c-}_{(g,p)}$, $\Phi^{-\xi}_{(g,p)}$ and $\Phi^{--}_{(g,p)}$ are annihilated by both $b_0$ and $\eta_0$.

$$\begin{cases} b_0\Phi_{(g,p)} = 0 \iff \Phi^{c-}_{(g,p)} = \Phi^{c\xi}_{(g,p)} = 0\,, \\ \xi_0\Phi_{(g,p)} = 0 \iff \Phi^{c-}_{(g,p)} = \Phi^{--}_{(g,p)} = 0\,, \\ b_0\xi_0\Phi_{(g,p)} = 0 \iff \Phi^{c-}_{(g,p)} = 0\,. \end{cases}$$

$\boxed{S_0}$ $$S_0 = -\frac{1}{2}\int \Phi_{(0,0)}(Q\eta_0\Phi_{(0,0)}) \quad \text{with} \quad \begin{bmatrix} b_0 \\ \xi_0 \end{bmatrix} \Phi_{(0,0)} = 0$$

$$\Longrightarrow S_0 = -\frac{1}{2}\int (\xi_0\Phi^{-\xi}_{(0,0)})Q\eta_0(\xi_0\Phi^{-\xi}_{(0,0)}) = -\frac{1}{2}\int \Phi^{-\xi}_{(0,0)}(c_0L_0)\xi_0\Phi^{-\xi}_{(0,0)}$$

We need one $c_0$ and one $\xi_0$ to obtain nonzero $\int(\cdots)$ .

cf. $$S_0^{\text{Witten}} = -\frac{1}{2}\int_{\mathrm{S}} \Psi_{(1,-1)}(c_0L_0)\Psi_{(1,-1)} \,.$$

Correspondence: $\Phi^{-\xi}_{(0,0)} \longleftrightarrow \Psi_{(1,-1)}$ $\quad \left(\int \xi_0(\cdots) = \int_{\mathrm{S}}(\cdots)\right)$

$\boxed{S_1}$ $$S_1 = -\int \Phi_{(2,-1)} \left(Q\Phi_{(-1,0)} + \eta_0 \Phi_{(-1,1)}\right)$$

with $$\begin{bmatrix} b_0 & 0 \\ \xi_0 & 0 \\ 0 & \xi_0 \end{bmatrix} \begin{bmatrix} \Phi_{(-1,0)} \\ \Phi_{(-1,1)} \end{bmatrix} = 0\,, \quad b_0 \xi_0 \Phi_{(2,-1)} = 0\,.$$

$$\Longrightarrow S_1 = -\int \left(c_0 \xi_0 \Phi^{c\xi}_{(2,-1)} + \xi_0 \Phi^{-\xi}_{(2,-1)} + \Phi^{--}_{(2,-1)}\right) Q \left(\xi_0 \Phi^{-\xi}_{(-1,0)}\right)$$
$$-\int \left(c_0 \xi_0 \Phi^{c\xi}_{(2,-1)}\right) \Phi^{-\xi}_{(-1,1)} + \int \left(\xi_0 \Phi^{-\xi}_{(2,-1)}\right) \left(c_0 \Phi^{c\xi}_{(-1,1)}\right).$$

$\Phi^{-\xi}_{(-1,1)}$ and $\Phi^{c\xi}_{(-1,1)}$ can be regarded as Lagrange multipliers which impose $\Phi^{c\xi}_{(2,-1)} = \Phi^{-\xi}_{(2,-1)} = 0\,.$

$$\Longrightarrow S_1 = -\int \Phi^{--}_{(2,-1)} Q\big(\xi_0 \Phi^{-\xi}_{(-1,0)}\big) = -\int \Phi^{--}_{(2,-1)} (c_0 L_0) \xi_0 \Phi^{-\xi}_{(-1,0)} \,.$$

cf. $$S_1^{\rm Witten} = -\int_{\rm S} \Psi_{(2,-1)} (c_0 L_0) \Psi_{(0,-1)} \,.$$

Correspondence: $\Phi^{-\xi}_{(-1,0)} \longleftrightarrow \Psi_{(0,-1)}\,, \quad \Phi^{--}_{(2,-1)} \longleftrightarrow \Psi_{(2,-1)}\,.$

Similarly we obtain

$$\Phi^{-\xi}_{(-2,0)} \longleftrightarrow \Psi_{(1,-1)}\,, \quad \Phi^{--}_{(3,-1)} \longleftrightarrow \Psi_{(3,-1)}\,, \quad \cdots \,.$$

The two gauge-fixed free actions correspond to each other.

# 4. Propagators

$\Phi_{(0,0)}$-Propagator $\mathbf{P}$

$\mathbf{P}$ is the inverse of $Q\eta_0$ in $S_0 = -\frac{1}{2}\langle\Phi_{(0,0)}|Q\eta_0|\Phi_{(0,0)}\rangle$

under the condition $\begin{bmatrix} b_0 \\ \xi_0 \end{bmatrix} |\Phi_{(0,0)}\rangle = 0 \quad \Big( \Longleftrightarrow \ \langle\Phi_{(0,0)}| \begin{bmatrix} b_0 & \xi_0 \end{bmatrix} = 0 \Big)$.

## $\mathbf{P}$ as the inverse of $Q\eta_0$

(1) $\mathbf{P}(Q\eta_0)$ acts as $1$ on the restricted subspace of $|\Phi_{(0,0)}\rangle$:

$$\mathbf{P}(Q\eta_0) = 1 + \mathbf{M_{1,2}} \begin{bmatrix} b_0 \\ \xi_0 \end{bmatrix} \cong 1 \qquad (\mathbf{M_{1,2}} \text{ is some } 1 \times 2 \text{ matrix}).$$

(2) $(Q\eta_0)\mathbf{P}$ acts as $1$ on the restricted subspace of $\langle\Phi_{(0,0)}|$:

$$(Q\eta_0)\mathbf{P} = 1 + \begin{bmatrix} b_0 & \xi_0 \end{bmatrix} \mathbf{M_{2,1}} \cong 1 \qquad (\mathbf{M_{2,1}} \text{ is some } 2 \times 1 \text{ matrix}).$$

If $\mathrm{bpz}(\mathbf{P}) = \mathbf{P}$, condition (2) follows from (1).

$\Longrightarrow$ We consider BPZ-even $\mathbf{P}$.

Answer: $$\mathbf{P} = \frac{\xi_0 b_0}{L_0}$$

Check

$$\mathbf{P}(Q\eta_0) = 1 + \begin{bmatrix} -\xi_0\eta_0\frac{Q}{L_0} & -\eta_0 \end{bmatrix} \begin{bmatrix} b_0 \\ \xi_0 \end{bmatrix}.$$

## Ghost Propagators

First we rewrite the action:

$$S_1 = -\langle\Phi_{2,-1}|\left(Q\,|\Phi_{(-1,0)}\rangle + \eta_0\,|\Phi_{(-1,1)}\rangle\right)$$

$$= -\frac{1}{2}\left[\,\langle\Phi_{(-1,0)}| \quad \langle\Phi_{(-1,1)}| \quad \langle\Phi_{(2,-1)}|\,\right]\begin{bmatrix} 0 & 0 & Q \\ 0 & 0 & \eta_0 \\ Q & \eta_0 & 0 \end{bmatrix}\begin{bmatrix} |\Phi_{(-1,0)}\rangle \\ |\Phi_{(-1,1)}\rangle \\ |\Phi_{(2,-1)}\rangle \end{bmatrix}$$

with

$$\begin{bmatrix} b_0 & 0 \\ \xi_0 & 0 \\ 0 & \xi_0 \end{bmatrix}\begin{bmatrix} |\Phi_{(-1,0)}\rangle \\ |\Phi_{(-1,1)}\rangle \end{bmatrix} = b_0\xi_0\,|\Phi_{(2,-1)}\rangle = 0\,.$$

We consider the BPZ-even inverse $\begin{bmatrix} \mathbf{0_2} & \mathrm{bpz}(\mathbf{P_{1,2}}) \\ \mathbf{P_{1,2}} & \mathbf{0_1} \end{bmatrix}$.

The propagator $\mathbf{P_{1,2}}$ is a $1 \times 2$ matrix.

Definition of the BPZ conjugate of a matrix $\mathbf{M} = (M_{ij})$:

$$\Big(\big(\mathrm{bpz}(\mathbf{M})\big)_{ij}\Big) := \mathrm{bpz}(M_{ji}) \,.$$

If the matrices $\mathbf{A}$ and $\mathbf{B}$ have definite Grassmann parities, we have

$$\mathrm{bpz}(\mathbf{AB}) = (-1)^{\epsilon(\mathbf{A})\epsilon(\mathbf{B})}\mathrm{bpz}(\mathbf{B})\,\mathrm{bpz}(\mathbf{A}) \,.$$

Since $\begin{bmatrix} \mathbf{0_2} & \mathrm{bpz}(\mathbf{P_{1,2}}) \\ \mathbf{P_{1,2}} & \mathbf{0_1} \end{bmatrix} \begin{bmatrix} 0 & 0 & Q \\ 0 & 0 & \eta_0 \\ Q & \eta_0 & 0 \end{bmatrix} \cong 1$, the following equations should hold.

$$\mathrm{bpz}(\mathbf{P_{1,2}}) \begin{bmatrix} Q & \eta_0 \end{bmatrix} = \mathbf{1_2} + \mathbf{M_{2,3}} \begin{bmatrix} b_0 & 0 \\ \xi_0 & 0 \\ 0 & \xi_0 \end{bmatrix} \cong \mathbf{1_2} \quad \text{on} \quad \begin{bmatrix} |\Phi_{(-1,0)}\rangle \\ |\Phi_{(-1,1)}\rangle \end{bmatrix},$$

$$\mathbf{P_{1,2}} \begin{bmatrix} Q \\ \eta_0 \end{bmatrix} = 1 + \mathbf{M_{1,1}}\, b_0 \xi_0 \cong 1 \quad \text{on} \quad |\Phi_{(2,-1)}\rangle \,.$$

We can solve the above equations component by component.

Result: $\mathbf{P_{1,2}} = \begin{bmatrix} A & B \end{bmatrix} := \begin{bmatrix} \frac{1}{L_0}\eta_0\xi_0 b_0 & \left(1 - \frac{1}{L_0} X_0 b_0 \eta_0\right) \xi_0 \end{bmatrix}$ .

The zero mode $X_0$ of the PCO appears.

## Propagation between $\Phi_{(-n,*)}$ and $\Phi_{(n+1,\star)}$

$$\mathbf{P_{n,n+1}} = \begin{bmatrix} A & B & (-X_0)B & (-X_0)^2B & \cdots & (-X_0)^{n-1}B \\ 0 & 0 & \xi_0 & (-X_0)\xi_0 & \cdots & (-X_0)^{n-2}\xi_0 \\ 0 & 0 & 0 & \xi_0 & \cdots & (-X_0)^{n-3}\xi_0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \xi_0 \end{bmatrix}.$$

The action does not include PCOs, but the propagators do.

# 5. Another Gauge-Fixing Condition

Instead of $\xi_0$, we can use $d_0$, the zero mode of $d := [Q,\, b\xi]$.
($d_0$ is nothing but $\widetilde{G}_0^-$ in the $N=2$ superconformal algebra.)

Key relations

$$\begin{cases} b_0^2 = d_0^2 = \{b_0,\, d_0\} = 0\,, \\ \left\{Q,\, \dfrac{b_0}{L_0}\right\} = \left\{\eta_0,\, \dfrac{d_0}{L_0}\right\} = 1 \\ \{\eta_0,\, b_0\} = \{Q,\, d_0\} = 0\,. \end{cases} \qquad \underline{\text{cf.}} \quad \begin{cases} \xi_0^2 = \{b_0,\, \xi_0\} = 0\,, \\ \{\eta_0,\, \xi_0\} = 1\,, \\ \{Q,\, \xi_0\} = X_0\,. \end{cases}$$

# $b_0$-$d_0$ Gauge

Just replace $\xi_0$ with $d_0$.

$$\begin{bmatrix} b_0 \\ \xi_0 \end{bmatrix} \Phi_{(0,0)} = \begin{bmatrix} b_0 & 0 \\ \xi_0 & 0 \\ 0 & \xi_0 \end{bmatrix} \begin{bmatrix} \Phi_{(-1,0)} \\ \Phi_{(-1,1)} \end{bmatrix} = \begin{bmatrix} b_0 & 0 & 0 \\ \xi_0 & 0 & 0 \\ 0 & \xi_0 & 0 \\ 0 & 0 & \xi_0 \end{bmatrix} \begin{bmatrix} \Phi_{(-2,0)} \\ \Phi_{(-2,1)} \\ \Phi_{(-2,2)} \end{bmatrix} = \cdots = 0\,,$$

$$b_0\xi_0\Phi_{(2,-1)} = \begin{bmatrix} b_0\xi_0 & 0 \\ 0 & \xi_0 \end{bmatrix} \begin{bmatrix} \Phi_{(3,-1)} \\ \Phi_{(3,-2)} \end{bmatrix} = \begin{bmatrix} b_0\xi_0 & 0 & 0 \\ 0 & \xi_0 & 0 \\ 0 & 0 & \xi_0 \end{bmatrix} \begin{bmatrix} \Phi_{(4,-1)} \\ \Phi_{(4,-2)} \\ \Phi_{(4,-3)} \end{bmatrix} = \cdots = 0\,.$$

# $b_0$-$d_0$ Gauge

Just replace $\xi_0$ with $d_0$.

$$\begin{bmatrix} b_0 \\ d_0 \end{bmatrix} \Phi_{(0,0)} = \begin{bmatrix} b_0 & 0 \\ d_0 & 0 \\ 0 & d_0 \end{bmatrix} \begin{bmatrix} \Phi_{(-1,0)} \\ \Phi_{(-1,1)} \end{bmatrix} = \begin{bmatrix} b_0 & 0 & 0 \\ d_0 & 0 & 0 \\ 0 & d_0 & 0 \\ 0 & 0 & d_0 \end{bmatrix} \begin{bmatrix} \Phi_{(-2,0)} \\ \Phi_{(-2,1)} \\ \Phi_{(-2,2)} \end{bmatrix} = \cdots = 0\,,$$

$$b_0 d_0 \Phi_{(2,-1)} = \begin{bmatrix} b_0 d_0 & 0 \\ 0 & d_0 \end{bmatrix} \begin{bmatrix} \Phi_{(3,-1)} \\ \Phi_{(3,-2)} \end{bmatrix} = \begin{bmatrix} b_0 d_0 & 0 & 0 \\ 0 & d_0 & 0 \\ 0 & 0 & d_0 \end{bmatrix} \begin{bmatrix} \Phi_{(4,-1)} \\ \Phi_{(4,-2)} \\ \Phi_{(4,-3)} \end{bmatrix} = \cdots = 0\,.$$

# Propagators in $b_0$-$d_0$ Gauge

Replacement: $\xi_0 \to \frac{d_0}{L_0}, \quad X_0 \to 0$.

$$\mathbf{P} = \frac{1}{L_0}\xi_0 b_0 \,,$$

$$\mathbf{P_{1,2}} = \begin{bmatrix} A & B \end{bmatrix} := \begin{bmatrix} \frac{1}{L_0}\eta_0\xi_0 b_0 & \left(1 - \frac{1}{L_0}X_0 b_0 \eta_0\right)\xi_0 \end{bmatrix},$$

$$\mathbf{P_{n,n+1}} = \begin{bmatrix} A & B & (-X_0)B & (-X_0)^2 B & \cdots & (-X_0)^{n-1}B \\ 0 & 0 & \xi_0 & (-X_0)\xi_0 & \cdots & (-X_0)^{n-2}\xi_0 \\ 0 & 0 & 0 & \xi_0 & \cdots & (-X_0)^{n-3}\xi_0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \xi_0 \end{bmatrix}.$$

# Propagators in $b_0$-$d_0$ Gauge

Replacement: $\xi_0 \to \frac{d_0}{L_0}, \quad X_0 \to 0.$

$$\mathbf{P} = \frac{1}{L_0^2} d_0 b_0 \,,$$

$$\mathbf{P_{1,2}} = \begin{bmatrix} A & B \end{bmatrix} := \begin{bmatrix} \frac{1}{L_0^2}\eta_0 d_0 b_0 & \frac{1}{L_0} d_0 \end{bmatrix},$$

$$\mathbf{P_{n,n+1}} = \begin{bmatrix} \frac{1}{L_0^2}\eta_0 d_0 b_0 & \frac{1}{L_0} d_0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{L_0} d_0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \frac{1}{L_0} d_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{L_0} d_0 \end{bmatrix}.$$

# 6. Summary and Discussions

Summary

- The free WZW-like action in the $b_0$-$\xi_0$ gauge corresponds to the free Witten action in the Siegel gauge.

- In the $b_0$-$\xi_0$ gauge, the zero mode $X_0$ appears in the propagators.

- $b_0$-$d_0$ gauge simplifies the propagators.

## Discussions One-parameter extension $(\zeta_0 = \xi_0\,,\ d_0)$

$$\begin{bmatrix} b_0 \\ \zeta_0 \end{bmatrix} \Phi_{(0,0)} = \begin{bmatrix} b_0 & 0 \\ \zeta_0 & \alpha b_0 \\ 0 & \zeta_0 \end{bmatrix} \begin{bmatrix} \Phi_{(-1,0)} \\ \Phi_{(-1,1)} \end{bmatrix} = \begin{bmatrix} b_0 & 0 & 0 \\ \zeta_0 & \alpha b_0 & 0 \\ 0 & \zeta_0 & \alpha b_0 \\ 0 & 0 & \zeta_0 \end{bmatrix} \begin{bmatrix} \Phi_{(-2,0)} \\ \Phi_{(-2,1)} \\ \Phi_{(-2,2)} \end{bmatrix} = \cdots = 0\,,$$

$$b_0\zeta_0\Phi_{(2,-1)} = \begin{bmatrix} b_0\zeta_0 & 0 \\ \alpha b_0 & \zeta_0 \end{bmatrix} \begin{bmatrix} \Phi_{(3,-1)} \\ \Phi_{(3,-2)} \end{bmatrix} = \begin{bmatrix} b_0\zeta_0 & 0 & 0 \\ \alpha b_0 & \zeta_0 & 0 \\ 0 & \alpha b_0 & \zeta_0 \end{bmatrix} \begin{bmatrix} \Phi_{(4,-1)} \\ \Phi_{(4,-2)} \\ \Phi_{(4,-3)} \end{bmatrix} = \cdots = 0\,.$$

So far we have seen two cases: $(\zeta_0,\ \alpha) = (\xi_0,\ 0)\,,\ (d_0,\ 0)\,.$

# Symmetric $b_0$-$d_0$ Gauge

We consider the symmetric $b_0$-$d_0$ gauge.

$$\begin{bmatrix} b_0 \\ d_0 \end{bmatrix} \Phi_{(0,0)} = \begin{bmatrix} b_0 & 0 \\ d_0 & b_0 \\ 0 & d_0 \end{bmatrix} \begin{bmatrix} \Phi_{(-1,0)} \\ \Phi_{(-1,1)} \end{bmatrix} = \begin{bmatrix} b_0 & 0 & 0 \\ d_0 & b_0 & 0 \\ 0 & d_0 & b_0 \\ 0 & 0 & d_0 \end{bmatrix} \begin{bmatrix} \Phi_{(-2,0)} \\ \Phi_{(-2,1)} \\ \Phi_{(-2,2)} \end{bmatrix} = \cdots = 0\,,$$

$$b_0 d_0 \Phi_{(2,-1)} = \begin{bmatrix} b_0 & d_0 \end{bmatrix} \begin{bmatrix} \Phi_{(3,-1)} \\ \Phi_{(3,-2)} \end{bmatrix} = \begin{bmatrix} b_0 & d_0 & 0 \\ 0 & b_0 & d_0 \end{bmatrix} \begin{bmatrix} \Phi_{(4,-1)} \\ \Phi_{(4,-2)} \\ \Phi_{(4,-3)} \end{bmatrix} = \cdots = 0\,.$$

# Propagators in the Symmetric $b_0$-$d_0$ Gauge

$$\mathbf{P} = \frac{1}{L_0^2} d_0 b_0 \,,$$

$$\mathbf{P_{1,2}} = \begin{bmatrix} A & B \end{bmatrix} := \begin{bmatrix} \frac{b_0}{2L_0}\left(1 + \frac{\eta_0 d_0}{L_0}\right) & \frac{d_0}{2L_0}\left(1 + \frac{Q b_0}{L_0}\right) \end{bmatrix} \,,$$

$$\mathbf{P_{n,n+1}} = \begin{bmatrix} A & \frac{d_0}{2L_0} & & & \text{\Huge 0} \\ & \frac{b_0}{2L_0} & \ddots & & \\ & & \ddots & \frac{d_0}{2L_0} & \\ \text{\Huge 0} & & & \frac{b_0}{2L_0} & B \end{bmatrix}$$

## Further Extension

The gauge-fixed action $S$ can be rewritten as

$$S = S_0 - \sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\int \Phi_{(n+1,-m-1)}\big(Q\Phi_{(-n,m)} + \eta_0\Phi_{(-n,m+1)}\big)$$

$$= S_0 - \frac{1}{2}\sum_{n=1}^{\infty}\int \begin{bmatrix}\mathbf{\Phi}^{\mathrm{T}}_{\mathbf{-n}} & \mathbf{\Phi}^{\mathrm{T}}_{\mathbf{n+1}}\end{bmatrix} \begin{bmatrix}\mathbf{0_{n+1}} & \mathbf{Q_{n+1,n}} \\ -\mathrm{bpz}(\mathbf{Q_{n,n+1}}) & \mathbf{0_n}\end{bmatrix} \begin{bmatrix}\mathbf{\Phi_{-n}} \\ \mathbf{\Phi_{n+1}}\end{bmatrix}.$$

$$\mathbf{\Phi_{-n}} = \left.\begin{bmatrix}\Phi_{(-n,\,0)} \\ \vdots \\ \Phi_{(-n,\,n)}\end{bmatrix}\right\} n+1\,, \quad \mathbf{\Phi_{n+1}} = \left.\begin{bmatrix}\Phi_{(n+1,\,-1)} \\ \vdots \\ \Phi_{(n+1,\,-n)}\end{bmatrix}\right\} n\,,$$

$$\mathbf{Q_{n+1,n}} = \underbrace{\left.\begin{bmatrix} Q & & \text{\huge 0} \\ \eta_0 & \ddots & \\ & \ddots & Q \\ \text{\huge 0} & & \eta_0 \end{bmatrix}\right\}}_{n} n+1\,, \quad \mathbf{Q_{n,n+1}} = \underbrace{\left.\begin{bmatrix} Q & \eta_0 & & \text{\huge 0} \\ & \ddots & \ddots & \\ \text{\huge 0} & & Q & \eta_0 \end{bmatrix}\right\}}_{n+1} n\,.$$

The propagators we have obtained satisfies

$$\mathbf{P_{n+1,n+2}}\,\mathbf{Q_{n+2,n+1}} + \mathbf{Q_{n+1,n}}\,\mathbf{P_{n,n+1}} = \mathbf{1_{n+1}}\,.$$

Using this kind of relation, we can easily generalize our result.

ex. Extension similar to linear $b$-gauges

## Notes on the interacting theory

Witten's SSFT $\begin{cases} \text{includes } X(z_{\text{mid}}) \,. \\ \text{is cubic.} \end{cases}$

In the WZW-like SSFT

Is there a condition which $\begin{cases} \text{produces } X(z_{\text{mid}}) \,? \\ \text{makes the gauge-fixed action cubic?} \end{cases}$

If such a gauge-fixing condition exist, the WZW-like SSFT may be regarded as the regularization of Witten's.

Can we use $\xi(z_{\rm mid})\,\Phi_{(g,p)} = 0$ in stead of $\xi_0\,\Phi_{(g,p)} = 0$ ?

$$\left(\begin{array}{l} \xi(z_{\rm mid})\,\Phi_{(g,p)} = 0 \implies \Phi_{(g,p)} = \xi(z_{\rm mid})\eta_0\,\Phi_{(g,p)}\,, \\ Q\big(\xi(z_{\rm mid})\eta_0\,\Phi_{(g,p)}\big) = \big(X(z_{\rm mid}) - \xi(z_{\rm mid})Q\big)\eta_0\Phi_{(g,p)}\,, \\ \big(\xi(z_{\rm mid})\eta_0\,\Phi_{(g,p)}\big)\cdots\big(\xi(z_{\rm mid})\eta_0\,\Phi_{(g',p')}\big) = 0\,? \end{array}\right)$$

$\Longrightarrow$ No. The above condition is singular: $X(z_{\rm mid})\xi(z_{\rm mid}) = \infty$ .
We have to regularize it.

There may not exist a gauge-fixing condition which connect the two interacting theories.

## The way to determine $\mathbf{P_{1,2}}$ in the $b_0$-$\xi_0$ Gauge (with $\alpha = 0$)

$$\mathbf{P_{1,2}} \begin{bmatrix} Q \\ \eta_0 \end{bmatrix} = 1 + \mathbf{M_{1,1}}\, b_0\xi_0 \quad \Longleftrightarrow \quad AQ + B\eta_0 = 1 + \mathbf{M_{1,1}}\, b_0\xi_0$$

To obtain "1," we use $\{Q,\, \frac{b_0}{L_0}\} = \{\eta_0,\, \xi_0\} = 1$ .

- Assume $A = \mathcal{O}\frac{b_0}{L_0}$ .
- We need $b_0\xi_0$ . $\Longrightarrow$ Assume $\mathcal{O} = \eta_0\xi_0 \quad \leftarrow (g,p) = (0,0)$ .

$$AQ = \eta_0\xi_0\frac{b_0 Q}{L_0} = \eta_0\xi_0(1 - \frac{Q b_0}{L_0}) = \underline{\eta_0\xi_0} - \frac{1}{L_0}\eta_0 X_0 b_0 + \frac{1}{L_0}\eta_0 Q \xi_0 b_0 \,.$$

(We have moved $\xi_0$ to the right to obtain $b_0\xi_0$ .)

$$AQ = \underline{\eta_0 \xi_0} + \frac{1}{L_0} X_0 b_0 \eta_0 - \frac{1}{L_0} \eta_0 Q b_0 \xi_0$$

- To get $1$ and to cancel $X_0$ , $B\eta_0$ should be

$$B\eta_0 = \underline{\xi_0 \eta_0} - \frac{1}{L_0} X_0 b_0 \eta_0 = \begin{cases} \left(\xi_0 - \dfrac{1}{L_0} X_0 b_0\right) \eta_0 \,, \\ \left(\xi_0 - \dfrac{1}{L_0} X_0 b_0 \eta_0 \xi_0\right) \eta_0 \,. \end{cases}$$

$\Longrightarrow B = \xi_0 - \frac{1}{L_0} X_0 b_0$ ?? $\quad B = \xi_0 - \frac{1}{L_0} X_0 b_0 \eta_0 \xi_0$ ??

$\Longrightarrow$ The second $B$ is correct.

$$\Longrightarrow \mathbf{P_{1,2}} = \begin{bmatrix} A & B \end{bmatrix} = \begin{bmatrix} \frac{1}{L_0} \eta_0 \xi_0 b_0 & \left(1 - \frac{1}{L_0} X_0 b_0 \eta_0\right) \xi_0 \end{bmatrix}.$$